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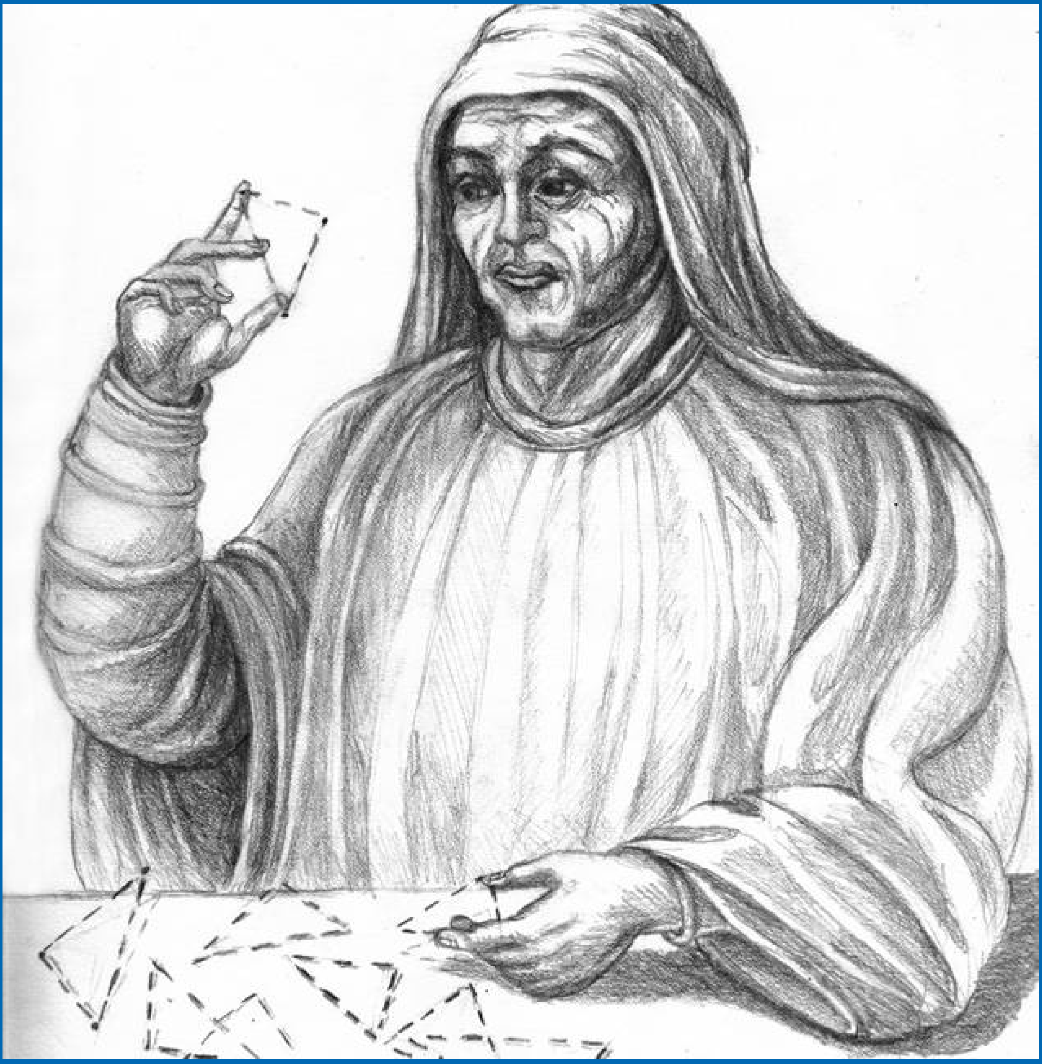
MAA

MATHEMATICS MAGAZINE



Alcuin's triangles

- Indivisibles and infinitesimals
- Antiderivatives without integrals
- Problems, problems, problems

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MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

How did Alcuin of York get on our cover? You can read the story on page 264.

Our first article—by Maureen Carroll, Steven Dougherty, and David Perkins, starting on the facing page—is about infinitesimals. These were an essential tool in the 17th century, when Torricelli and Roberval used them for some very clever calculations of areas. Nowadays we use integrals, because we find them to be more rigorous. But this article shows how the early methods can be strengthened using a rigorous, modern treatment of infinitesimals.

Integrals come in for some more disrespect in Adam Besenyei's note. We all know that continuous functions have antiderivatives, and we can prove it by construction using Riemann integrals. But, following Lebesgue, Besenyei proves the same result without integrals, and then derives basic results about integration from the existence of antiderivatives.

Harold Parks proves a famous limit formula involving the volume of the unit n-ball—again, without integrals. If, after all this, you still want to compute integrals, then check out Frank Sandomierski's note on error estimates.

Antonio Oller-Marcén's note combines number theory and geometry. I was pleased to see his citation of a paper from an early volume (1943) of this MAGAZINE. In those days it was called the *National Mathematics Magazine*; the MAA was supporting it modestly but had not yet become its publisher.

The note by Beauregard and Dobrushkin is about generating functions, and how they can be used to find a closed formula for the (finite) partial sums of an integer sequence. (It is the Alcuin sequence.)

James Harper offers us another bit of mathematical technology, for mass-producing integer solutions to the equation $A^3 + B^3 + C^3 = D^3$.

Did you know that the Problems section is the most frequently downloaded part of the MAGAZINE? Problems teach, entertain, and inspire us, and communities form around them. This issue's reports of the USAMO, the USAJMO, and the IMO show the problem-solving community at its best. These programs succeed in large measure because of the talents of MAA committee members and staff.

Congratulations to this year's recipients of the Allendoerfer Award. They are Khristo Boyadzhiev, for his article on Stirling numbers, and Adrian Rice and Bud Brown, for their article on elliptic curves. Both articles appeared in this MAGAZINE in 2012.

Walter Stromquist, Editor

ARTICLES

Indivisibles, Infinitesimals and a Tale of Seventeenth-Century Mathematics

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Imagine revolving an infinitely long region of infinite area around an axis, only to obtain an object of finite volume. Now imagine computing this finite volume by decomposing it into infinitely many two-dimensional objects and summing their areas. This is precisely what Evangelista Torricelli (1608–1647) did, without the advantage of calculus. His reasoning, while highly intuitive, was not rigorous. It sparked imitation in some (including, perhaps, plagiarism) and rebuke from others. The ensuing debate was part of the rich tapestry of mathematics in the seventeenth century.

The seventeenth century is remarkable for the analytic geometry of René Descartes (1596–1650) and Pierre de Fermat (1601–1665), and the calculus of Gottfried Leibniz (1646–1716) and Isaac Newton (1643–1727). These breakthroughs developed from a mathematical culture concerned with problems of measurement: area, volume, time and distance. The mathematicians who worked just prior to the first publications on calculus tackled very difficult problems of measurement without the benefit of coordinate geometry or the machinery of calculus. Their solutions were often non-rigorous. Yet despite the controversy over the admissibility of their techniques, we cannot help but marvel at their elegance.

In the first part of this paper, we consider two problems solved prior to calculus using intuitive but non-rigorous techniques—a volume problem solved by Torricelli, and an area problem solved by Gilles Personne de Roberval (1602–1675). We also consider an objection to their methods raised by Paul Guldin (1577–1643).

While these problems are easily translated into the language of calculus, the beauty of the original techniques can be lost in translation. In the second part of this paper, we attempt to restore the beauty of the techniques by presenting them within the context of a modern, rigorous method. We avoid much of the machinery of calculus by using infinitesimals. More precisely, we rely on the axiomatic development of infinitesimals introduced in the 1960s by Abraham Robinson (1918–1974). We use infinitesimals, but not the calculus results that arise from their development.

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Indivisibles

Finding the area of a region bounded by curves has its roots in the writings of Eudoxus (408–355 BC) and Archimedes (287–212 BC), who employed the method of exhaustion, approximating an irregular region by filling it with more and more small regions of known area. This idea was also applied to find volumes of irregular solids. While Archimedes had great success with this technique, the insight necessary for constructing such a solution made it difficult to imitate. This method of approximation went essentially unchanged until the beginning of the seventeenth century, when Johannes Kepler (1571–1630) used infinitely small geometric quantities to find volumes of solids of revolution. Kepler's success led the Italian mathematician Bonaventura Cavalieri (1598–1647) to present his method of *indivisibles* to compute areas and volumes in 1635. An indivisible lives one dimension below its environment, like a page in a book (if we allow the conceit of two-dimensional pages). In this way, a page is a slice of a book, and enough pages pressed together create the three-dimensional book. If a page were truly two-dimensional, we would not be able to divide it into two thinner pages; it would be an indivisible.

This metaphor takes us to shaky ground, mathematically. Democritus (ca. 460–370 BC), one of the earliest thinkers to propose this idea, warned of its paradoxes. Imagine slicing a sphere, like an orange, into infinitely thin circular slices; is the last slice a circle, or a point? If a point, then what of the slice next to it? If that slice is not a point, then what is its radius? If we reassemble the slices into the original sphere, will the surface look like tiny stairs?

Despite these philosophical concerns, the theory of indivisibles enjoyed some significant traction with mathematicians of the day. A typical argument using indivisibles followed this strategy: Deconstruct a three-dimensional volume into infinitely many two-dimensional slices, deform each slice into a new shape, and then reassemble the altered slices into a new three-dimensional volume that is equivalent to the original. (The same technique applies if we think of an area as being composed of infinitely many line segments.) This method produces the solutions to the two problems we consider in this paper.

Torricelli finds the volume of Gabriel's Horn

We begin with Torricelli, who used the method of indivisibles to find a volume that, upon first inspection, seems to be infinite. The object resembled a trumpet with an infinitely long neck, stretching upward forever. Thus, it has come to be known as Gabriel's Trumpet, after the angel of that name. The trumpet is one of those marvels of mathematics that reminds us how beautiful a simply defined object can be. It is an object formed by revolving a region of infinite area around an axis. The resulting solid has infinite surface area and finite volume.

The trumpet lies within a hyperbola that is revolved around the vertical axis. In FIGURE 1, we see the hyperbola $y = 1/x$ (for $x > 0$) and its mirror image $y = 1/|x|$ (for $x < 0$), which together define the side view profile of the trumpet's neck. Choose any point C along the positive horizontal axis and locate the corresponding point D on the hyperbola; then $CD = 1/OC$. The region rotated about the vertical axis is bounded by the hyperbola $y = 1/x$, the segment CD, and the horizontal axis. Segment CD passes through $C'D'$ upon revolving, so the trumpet is made finite in the horizontal direction. Its neck, however, extends up forever, narrowing toward G, where the angel Gabriel waits for the divine command to play.

Torricelli peeled away two-dimensional layers of the trumpet as we now describe. For each point E between O and C, locate the corresponding point F on the hyperbola.

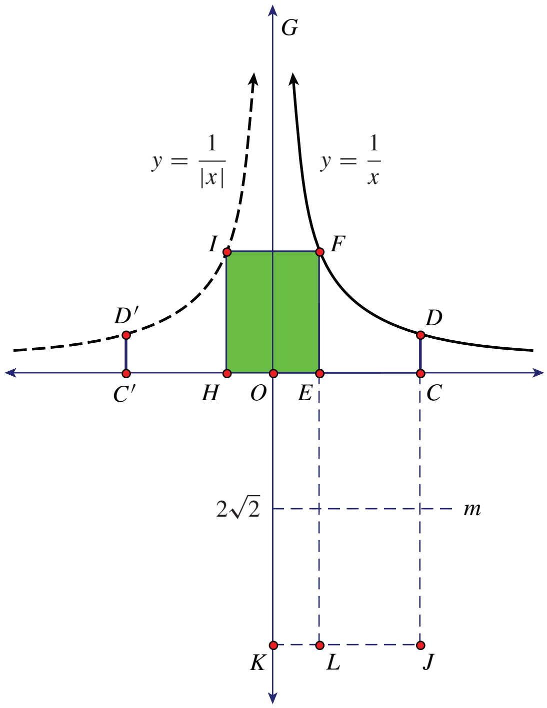

Figure 1 Torricelli's "trumpet" results from revolving region *FDCOG* around the vertical axis

Revolve EF around the vertical axis, creating a cylindrical shell with side view $EFIH$. Cut and unroll the shell into a flat rectangle. The area of this rectangle is its height (EF) times its width (the circumference $2\pi(OE)$ of the cylinder's base). Because $EF = 1/OE$, the area of the rectangle is simply 2π.

Notice that *every* such cylinder therefore occupies the two-dimensional area 2π, irrespective of how we choose point E. This sort of symmetry must have convinced Torricelli that he was indeed viewing the horn in a canonical manner. The rectangles produced by the unrolled cylinders have varying widths and heights, making them difficult to reassemble, so Torricelli exploited the fact that each unrolled cylinder occupies the same area, 2π, as every other. After deforming each rectangle, like pizza dough, into a circle of radius $\sqrt{2}$, Torricelli stacked the circles in such a way that the correspondence between rectangles and circles is undebatable. (While these rectangles could be reshaped into any region of area 2π, Torricelli must have found the circles to be a natural fit given the appearance of π and the constructibility of a circle of area 2π.)

In FIGURE 1, rectangle $OCJK$ with height $2\sqrt{2}$ is a side view of the cylinder (having axis m) that Torricelli fashions from his circles. Segment EL is the side view of the circle created by deforming the cylinder that corresponds to the point E. So as E ranges from C to O, each resulting cylinder matches with one circle in the cylinder. To help visualize these three-dimensional objects, the solid cylinder formed from the stacked circles is shown along with the hollow cylinder at point E in FIGURE 2. Thus, Torricelli claimed that the apparently infinite volume of the trumpet equals the clearly finite volume of the cylinder with axis m, $2\pi(OC)$.

Although he believed that his argument via indivisibles settled the matter, Torricelli also proved the same result with straightforward geometry [**2**]. Today, using calculus,

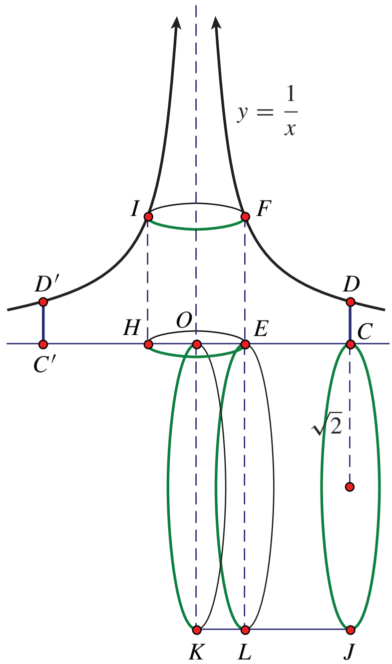

Figure 2 A cylindrical slice of Gabriel's Horn at point E and Torricelli's constructed cylinder

we may prove his result with the cylindrical shell method by calculating an integral of the form

$$2\pi \int_0^C x \cdot \frac{1}{x}\, dx,$$

a simple task, as long as one is willing to accept the intricate structure supporting calculus.

Roberval calculates the area under a cycloid

When Torricelli was born in 1608, the six-year-old Roberval was learning mathematics in France, over a thousand kilometers away. During their lives, the two men appealed to indivisibles with equal enthusiasm and shared their investigations through regular correspondence. Controversy ensued when Roberval claimed that Torricelli had plagiarized the discovery that we detail in this section; Torricelli died before the matter was resolved, but Roberval's accusation is not supported by modern scholars. Despite the quarrel, the discovery stands as a testament to the elegance of indivisibles.

If the circle with diameter AD in FIGURE 3 were to roll to its right while sitting on AG, the point A would track along the dashed curve AP, peaking at point C, and returning to the "ground" at point G. The curve $APCG$ is called a *cycloid*, and it is generated by a rolling circle; the point A acts like a pebble stuck in the tread of a rolling bicycle tire. The cycloid caught the fancy of mathematicians in the late 1600s, in part because it provided the surprising solution to the following two questions: Along what curve would a clock's pendulum swing so that the clock would keep perfect time no matter how far the pendulum traveled in one swing? What curve allows a ball placed on it to descend from one point to another in the fastest time? In each case, an inverted version of the cycloid shown in FIGURE 3 answers the question.

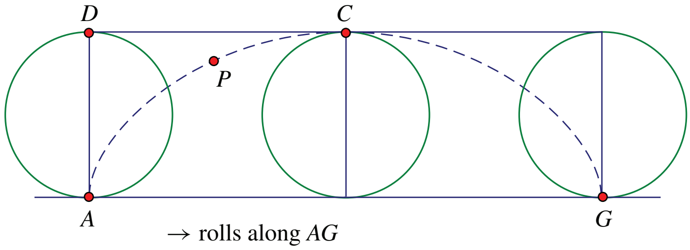

Figure 3 The cycloid *APCG* tracks the movement of *A* on the circle as it rolls from *A* to *G*

Galileo (1564–1642) approximated the area under a cycloid by building one of metal and weighing it against the metal circle that generated it. This experiment suggested to him that the area of the cycloid is about three times that of the generating circle. Roberval proved that it is *exactly* three times.

As Roberval did, we will find the area under the half-cycloid, and then double this result to find the desired area. FIGURE 4 shows the circle both in its original position and after rolling along half its circumference, forming a half-cycloid. It also shows an intermediate position of the circle, in which point A has moved up to A', and point E on the circle has become the point of contact labeled E' on the ground AB. Arcs AE and $A'E'$ both share the length of segment AE', so A has traveled vertically up the same distance that E has traveled vertically down. Construct line $A'E$ to meet AD perpendicularly at L, and place M on the line so that $LE = A'M$. As the circle rolls, this point M traces out a curve just like A does. (Roberval calls AMC the companion curve to the cycloid and finds it to be, using modern symbolism, $y = r - r\cos\frac{x}{r}$, where r is the radius of the generating circle [**1, 11**].) Since $LE = A'M$ at every moment, Roberval claimed that LE and $A'M$ would sweep through identical areas as point A travels towards point C to form the half-cycloid; hence, the semicircle AED has the same area as region $AA'CM$. This gives us one piece of the area under the cycloid, leaving us to find the area of region $AMCB$.

By symmetry, $AE'ML$ is a rectangle. So the path traveled by M on curve AMC is connected to the path traveled by L on AD. Imagine that the generating circle in FIGURE 4 is animated, and that this circle rolls on AB toward the right at a fixed rate. Point L acts as a reference point that tracks the vertical rise of A as the circle rolls. The ascent of L is not steady: It ascends more and more quickly until the circle has rolled halfway to B, and after that its ascent slows. But while the ascent of L is not steady, it *is* symmetric: The increase of its rate of ascent is mirrored by the decrease as a result of the symmetry of the circle. Therefore, curve AMC divides $ABCD$ in half.

The area of $ABCD$ is its height (the diameter of the circle) times its width (half of the circumference of the circle); that is, the area of $ABCD$ is twice the area of the

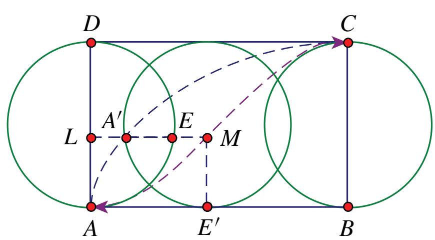

Figure 4 *A* travels the cycloid while *M* travels a path that mirrors the height of *A*

circle. With the area of both pieces resolved, we see the area of half-cycloid $AA'CB$ is half again the area of circle AED. Galileo's experimental estimate was correct: The area under a cycloid is exactly three times the area of the generating circle.

Indivisibles under attack

These discoveries, while celebrated, endured scrutiny by those who believed that mathematical truths should be supported by sound, unambiguous deduction. If a proof contains an approach that can be aimed at a different problem and "prove" an impossibility, then the approach needs repair. The Swiss mathematician Paul Guldin put Cavalieri's method to the test with an example that is as damaging as it is simple to understand [**1**]. Suppose that triangle $\triangle ADC$, shown in FIGURE 5, has $AD \neq DC$. Altitude BD, therefore, cuts the triangle's base AC into two unequal parts. Guldin proposed that we allow E to vary in its position on AB, as indicated by the arrows. Segment EF is perpendicular to the base, as is GH, and these two segments are equal in length. Thus, to each point E on AB, there corresponds one point H on BC such that $EF = GH$. Guldin, appealing to Cavalieri's method of indivisibles, claimed that if for each segment EF we get a corresponding segment GH, then the areas of triangles $\triangle ADB$ and $\triangle BDC$ must be equal. This absurd conclusion must rest, then, on a spurious claim. Guldin maintained that the only claim that could possibly be spurious in his argument was the one based on Cavalieri's method. Thus, the method is absurd.

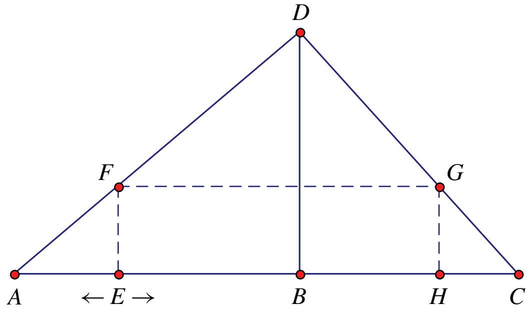

Figure 5 As E slides along segment AB, H mirrors its motion along BC

We shall show why the solutions of Torricelli and Roberval were correct, even though their technique was based on a false premise. It was, in fact, their superb intuition that saved them. With slight modification, rebuilding on the solid foundation of the infinitesimal, we produce the same results, making the work of these mathematicians even more impressive.

Infinitesimal calculus

Torricelli's and Roberval's use of indivisibles gives us a glimpse of the heuristic methods used to solve geometric problems before the invention of the calculus. To be fair, we should remember that the early practitioners of the calculus used infinitesimals in a similarly intuitive way, and lacked a rigorous axiomatic development. Both approaches drew their share of critics. Just as indivisibles faced criticism from Guldin, infinitesimals drew criticism from George Berkeley (1685–1753). While later mathematicians such as Leonhard Euler (1707–1783) took a step away from a reliance on heuristic arguments, the calculus would have to wait until the nineteenth century for Augustin Louis Cauchy (1789–1857) and Karl Weierstrass (1815–1897) to build a sound logical foundation for the two-centuries-old field. With Cauchy's initial work and Weierstrass's later refinements, calculus took a marked turn away from intuition-based arguments, and infinitesimals were abandoned in favor of limits. Nearly all calculus textbooks and courses still use Weierstrass's ϵ-δ definition of limits as the basis for further study.

While most mathematicians continued to build upon the work of Cauchy and Weierstrass, others tried to formalize the early work of the first practitioners of calculus. Most recently, Abraham Robinson (1918–1974) gave a rigorous development of calculus based on infinitesimals. Robinson gave the name *non-standard analysis* to this flavor of infinitesimal calculus. Every topic in a traditional calculus course can be taught using Robinson's foundation, and some argue that this approach is more natural for students, as it adheres more closely to the ideas that inspired Newton and Leibniz.

The geometric and intuitive solutions given by mathematicians such as Torricelli and Roberval can be revisited in light of Robinson's work. Here we will view their work through this lens. We will introduce only as much of the modern theory as we need. For further study of non-standard analysis, we recommend the original text by Robinson [**9**], a newer text by Henle and Kleinberg [**5**], and the first introductory calculus text to use this approach, by Keisler [**7**].

Non-standard analysis begins with the field of *hyperreals*, denoted by $\mathbb{HR}$. There are several ways to construct the hyperreals or to prove the existence of such a field, but for our purposes we need only know that the hyperreals form an ordered field in which the reals $\mathbb{R}$ form a designated subfield, and that $\mathbb{HR}$ contains at least one infinitesimal element. A positive *infinitesimal* is any element of the field that is larger than zero but smaller than any positive real number. We follow the notation of Henle [**5**] and denote a generic infinitesimal by $\circledcirc$. A negative infinitesimal is smaller than zero but larger than any negative real number. A positive *infinite* hyperreal is larger than any real number, while a negative infinite hyperreal is less than any real number. If $\circledcirc$ is a positive infinitesimal, then $-\circledcirc$ is negative infinitesimal, $1/\circledcirc$ is positive infinite, and $-1/\circledcirc$ is negative infinite. Thus, $\mathbb{HR}$ contains all of these types of numbers, in addition to the real numbers. A hyperreal is called *nonstandard* if it is not real.

If $\circledcirc \in \mathbb{HR}$ is a positive infinitesimal, then $\circledcirc < 2\circledcirc < 3\circledcirc < \cdots$ and $\circledcirc > \circledcirc^2 > \circledcirc^3 > \cdots > 0$. In general, when $0 < x < y$ and y/x is infinite, then x is said to be *infinitely smaller* than y and y is said to be *infinitely larger* than x. Since x/y is infinitesimal, we also say that x is infinitesimal with respect to y. For example, $\circledcirc^2$ is infinitesimal with respect to $\circledcirc$, and $\circledcirc^3$ is infinitesimal with respect to $\circledcirc^2$. (It is interesting to note that John Wallis (1616–1703), the mathematician who introduced the infinity symbol in 1655, not only treated $\frac{1}{\infty}$ as an infinitesimal, but performed arithmetic operations on $\frac{1}{\infty}$ as we do with $\circledcirc$ [**10**].)

If two hyperreal numbers b and c differ by an infinitesimal or zero, then we say that these numbers are infinitely close, and write $b \approx c$. For example, $4 \approx 4 + \circledcirc$, since $(4 + \circledcirc) - 4 = \circledcirc$. This relation forms an equivalence relation on the hyperreals. Each real number b is surrounded by a cloud of hyperreals that are infinitely close to b. For example, the cloud of hyperreals about 0 are precisely the infinitesimals.

Every finite hyperreal number is infinitely close to a unique real number. The real number that is infinitely close to finite $b \in \mathbb{HR}$ is called the standard part of b, denoted by $\boxed{b}$; for example, when $\circledcirc$ is an infinitesimal, we have $\boxed{\circledcirc} = 0$ and $\boxed{\pi - \circledcirc^2} = \pi$. (Note that an infinite hyperreal does not have a standard part.) This operation is a homomorphism that preserves order. In particular, when b and c are finite, we have $\boxed{b+c} = \boxed{b} + \boxed{c}$, $\boxed{bc} = \boxed{b}\,\boxed{c}$, and $\boxed{b} \leq \boxed{c}$ when $b \leq c$. As another example, when $\circledcirc$ is infinitesimal and r is a real number, $r\circledcirc$ is infinitesimal since $\boxed{r\circledcirc} = \boxed{r}\,\boxed{\circledcirc} = r \cdot 0 = 0$.

We now return to the work of Roberval, Torricelli, and Guldin, and augment the intuition of indivisibles with the rigor of Robinson's infinitesimals.

Addressing Guldin's objections

Guldin's objections called the validity of the method of indivisibles into question. Now, using infinitesimals, we can address Guldin's concern while remaining faithful to the intuitive method.

Given $\triangle ACD$, place A at the origin and AC along the x-axis. Let B be the orthogonal projection of D onto the x-axis. As shown in FIGURE 6, $A = (0, 0)$, $B = (b, 0)$, $C = (c, 0)$ and $D = (b, h)$, after assigning h as the altitude of $\triangle ACD$. Letting y_1 be the function for the line defined by AD and y_2 be that defined by CD, we have $y_1 = \frac{h}{b}x$ and $y_2 = \frac{h}{b-c}(x - c)$.

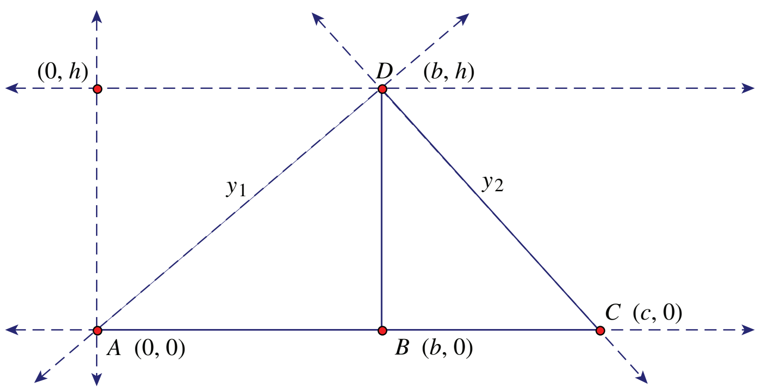

Figure 6 $\triangle ACD$ on the coordinate axes

Let $\circledcirc$ be a positive infinitesimal. We can divide segment AB—that is, the interval $[0, b]$—into infinitesimal pieces, each of width $b\circledcirc$. Letting $t_\alpha \in [0, b)$ be one of the points of the subdivision, the hyperreals $\{t_\alpha\}$ define an ordered partition of AB. We will designate the subdivision point immediately following a particular t_α, as $t_\beta = t_\alpha + b\circledcirc$.

For every $t_\alpha \in [0, b)$, there exists a corresponding $t'_\alpha \in [b, c)$ which can be determined in the following way, as shown in FIGURE 7. Construct the rectangle with lower left corner t_α, whose upper corners, F and G, lie on $\triangle ACD$. Then, $t'_\alpha \in [b, c)$ is the lower right corner of the rectangle. Using the formulas for y_1 and y_2, we find that $t'_\alpha = c + \frac{b-c}{b}\, t_\alpha$.

The hyperreals $\{t'_\alpha\}$ form an ordered partition of BC. There is a bijection of the partitions of AB and BC, but the distance between consecutive elements of the partition of BC is not necessarily the same as that of AB. Since $t_\beta = t_\alpha + b\circledcirc$, using y_1 and y_2 we have

$$t'_\beta = c + \frac{b-c}{b}\, t_\beta = c + \frac{b-c}{b}\,(t_\alpha + b\circledcirc).$$

Now we see that the partition of BC has subdivision width $t'_\alpha - t'_\beta = (c - b)\circledcirc$.

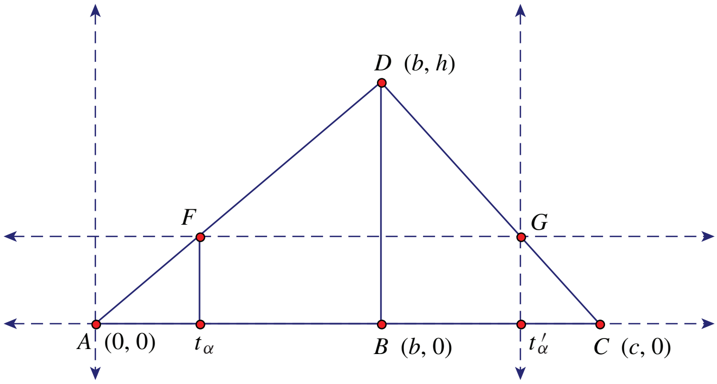

Figure 7 Corresponding partition points t_α and t'_α

Next, let's compare the infinitesimal areas of the corresponding trapezoids as shaded in FIGURE 8.

$$A_L = \text{Area(left trapezoid)} = \frac{1}{2} b \circledcirc [y_1(t_\alpha) + y_1(t_\beta)] = \frac{h\circledcirc}{2}[2t_\alpha + b\circledcirc]$$

$$A_R = \text{Area(right trapezoid)} = \frac{1}{2}(c-b) \circledcirc [y_2(t'_\alpha) + y_2(t'_\beta)]$$

$$= \frac{h \circledcirc (c-b)}{2b}[2t_\alpha + b\circledcirc].$$

Thus, the corresponding individual trapezoidal areas are equal if and only if $b = \frac{c}{2}$. Nevertheless, we must now consider the result of summing these infinitesimal areas. Let Ω_L represent the sum of all left trapezoids A_L, and Ω_R the sum of all right trapezoids A_R. To show that the sums of these left and right infinitesimal areas are not infinitely close when $b \neq c/2$, we will show that $\boxed{\Omega_L - \Omega_R} \neq 0$. First, we note that

$$\Omega_L = \frac{h}{2} \sum_\alpha \circledcirc[2t_\alpha + b\circledcirc] = \frac{h}{2} S,$$

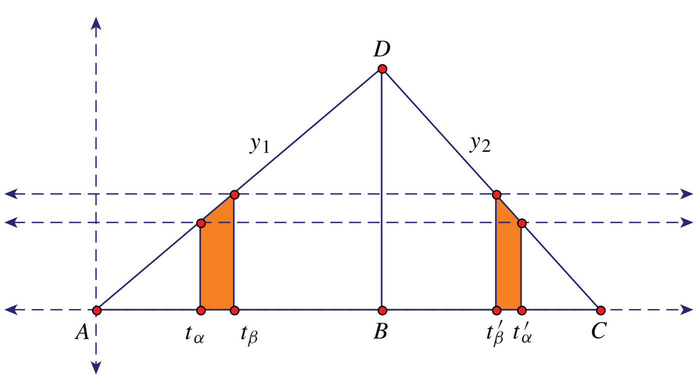

Figure 8 Corresponding trapezoidal areas

where $S = \sum_{\alpha} \circledcirc[2t_\alpha + b\circledcirc]$. Similarly,

$$\Omega_R = \frac{h(c-b)}{2b} \sum_{\alpha} \circledcirc[2t_\alpha + b\circledcirc] = \frac{h(c-b)}{2b} S.$$

Notice that these are infinite sums of hyperreals, and since $\triangle ABD$ and $\triangle CBD$ have finite positive real area, these sums are neither infinitesimal nor infinite. A formal treatment of series in the context of the hyperreals can be found in Chapter 5 of [**6**], Chapter 6 of [**4**], or Chapter 9 of [**5**]. For our purposes, we need only know that the distributive law applies in order to justify the factoring in these two equations.

Since the area of $\triangle ABD = \boxed{\Omega_L}$ is a positive real number, we know that S is not infinitesimal. Next, subtraction gives

$$\Omega_L - \Omega_R = \frac{h}{2} S - \frac{h(c-b)}{2b} S = \frac{h(2b-c)}{2b} S.$$

Neither of the factors on the right is infinitesimal, so neither is $\Omega_L - \Omega_R$. Hence, $\triangle ABD$ and $\triangle CBD$ are only equivalent in area when $\triangle ACD$ is isosceles, and Guldin's criticism has been addressed.

Roberval's method revisited

Roberval determined the area under the cycloid by finding the area under the half-cycloid *OPC* in FIGURE 9. Suppose that r is the radius of the generating circle *OAD*. Recall that, by Roberval's construction, we have: Segment *LA* is perpendicular to diameter *OD*, and points L, A, P and M all lie on the same perpendicular line, with M defined such that $LA = PM$. Also, as P travels the half-cycloid from O to C, M's path traces a curve *OMC* that divides the rectangle *OBCD* into congruent pieces. Thus, the area of the shaded region in FIGURE 10 is πr^2, since *OB* is half the circumference of the generating circle and *OD* is its diameter.

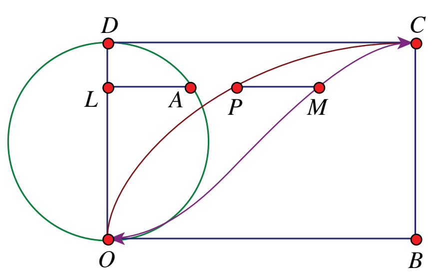

Figure 9 Cycloid with generating circle

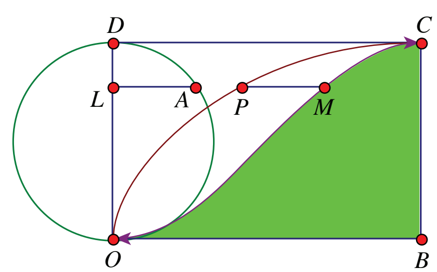

Figure 10 Part 1 of cycloidal area

Roberval then employed the theory of indivisibles to show that the area of the crescent-shaped shaded region in FIGURE 11 is equivalent to the area of semicircle *OAD*. Let's update his solution by using the hyperreals. Let $\circledcirc$ be a positive infinitesimal. Create a partition of *OD* consisting of subintervals of equal width $\circledcirc$. Suppose that L and L' are partition points separated by $\circledcirc$. We will consider two regions corresponding to a subinterval of the partition: the region $LL'A'A$ trapped within the generating circle, and $PP'M'M$ trapped between the cycloid *OPC* and the curve *OMC*, as shown

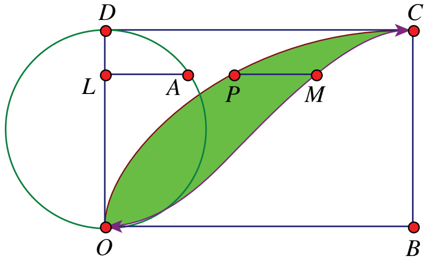

Figure 11 Part 2 of cycloidal area

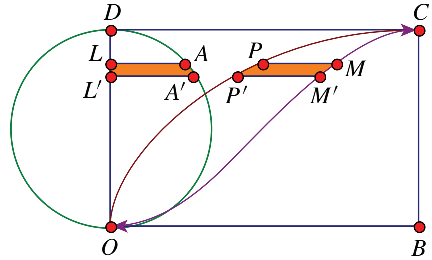

Figure 12 Corresponding regions

in FIGURE 12. Notice that $LA = PM$, $L'A' = P'M'$ and $LL' = \circledcirc$, by construction. We will compare the areas of these corresponding regions.

First, let's consider these regions as they exist in the lower half of the generating circle, as shown in FIGURE 13. In this figure, we have labeled $w = LA = PM$ and $w' = L'A' = P'M'$. Also notice that $w' < w$, since these regions are on the lower half of the generating circle. For Area_1, it is clear that

$$w'\circledcirc < \text{Area}_1 < w \circledcirc \,. \tag{1}$$

Figure 13 Corresponding regions in the lower half of rectangle *OBCD*

Within Area_2, drop a perpendicular from PM that intersects $P'M'$ at Q, as shown in FIGURE 14. By construction we have $PQ = \circledcirc$. Let's consider the length of segment $P'Q$. Recall that P' and P are points on the half-cycloid OPC. Since the cycloid is an increasing function, a non-infinitesimal positive change in input produces a non-infinitesimal positive change in output. Therefore, $P'Q$ is positive infinitesimal, since PQ is positive infinitesimal. Letting $P'Q = \circledcirc_2$, it is clear that

$$(w' - \circledcirc_2)\circledcirc < \text{Area}_2 < (w + \circledcirc_2)\circledcirc \,. \tag{2}$$

Also, since $\circledcirc_2$ is positive infinitesimal, Inequality (1) gives

$$(w' - \circledcirc_2)\circledcirc < w'\circledcirc < \text{Area}_1 < w\circledcirc < (w + \circledcirc_2)\circledcirc \,. \tag{3}$$

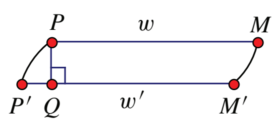

Figure 14 Area_2 bounded by cycloid and curve *OMC*

To find a bound on $w - w'$, we'll use the fact that A and A' are points on the lower half of the generating circle, as seen in FIGURE 15. The point S is the center of the circle. Thus, $SA = SA' = r$, where r is the radius of the generating circle, and

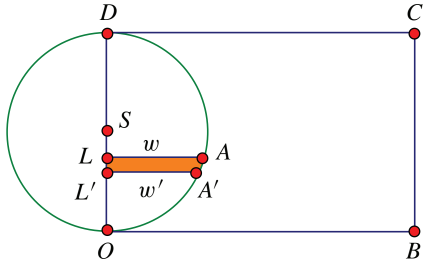

Figure 15 Area$_1$ within the generating circle

$LL' = ⊚$ by construction. Notice that $w^2 = r^2 - (LS)^2$ and $(w')^2 = r^2 - (L'S)^2 = r^2 - (LS + ⊚)^2$. Therefore,

$$\begin{aligned}(w - w')^2 &\leq (w - w')(w + w') \\ &= w^2 - (w')^2 \\ &= (r^2 - (LS)^2) - (r^2 - (LS + ⊚)^2) \\ &= 2(LS) ⊚ + ⊚^2 \\ &\leq 2r ⊚ + ⊚^2 \text{ since } LS \leq r.\end{aligned}$$

Thus, $|w - w'| \leq \sqrt{2r ⊚ + ⊚^2}$ for all such w and w'. The argument for the upper half of the generating circle is similar.

Let Ω_1 represent the sum of all Area$_1$ sections (producing the area of semicircle *OAD*) and Ω_2 the sum of the Area$_2$ sections (producing the area of region *OPCM*). We will show that the difference between these sums is infinitesimal, hence the real areas must be the same. Inequalities (2) and (3) give

$$|\Omega_1 - \Omega_2| \leq ⊚ \sum [w - w' + 2⊚_2] = ⊚ \sum [w - w'] + 2 \sum [⊚⊚_2],$$

where we are summing over all $\frac{2r}{⊚}$ pieces of the partition of *OD*. For the first summation, our bound on $w - w'$ allows us to simplify as follows:

$$⊚ \sum [w - w'] \leq ⊚ \cdot \left[\frac{2r}{⊚}\sqrt{⊚(2r + ⊚)}\right] = 2r\sqrt{⊚(2r + ⊚)},$$

an infinitesimal quantity. For the second summation, we only need to note that $⊚⊚_2$ is infinitesimal with respect to $⊚$. Since we are summing over $\frac{2r}{⊚}$ pieces of the partition, this sum is at most the same order as $⊚_2$, thus infinitesimal. We can now conclude that $\boxed{|\Omega_1 - \Omega_2|} = 0$, since both sums are infinitesimal.

Thus, the area of crescent *OPCM* is half the area of the generating circle. Combining this with the area of region *OMCB*, we see that the area of the half cycloid *OPC* is $\frac{3}{2}\pi r^2$. Therefore, the area of the cycloid is three times that of its generating circle.

Torricelli's method revisited

Let's revisit Gabriel's horn using infinitesimals. Torricelli considered the solid generated by rotating the region under the curve $y = \frac{1}{x}$ on the interval $(0, c]$, where $c > 0$, about the y-axis. Divide the hyperreal x-axis interval $(0, c]$ into segments of equal

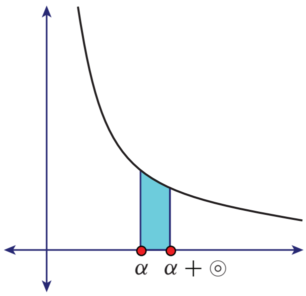

Figure 16 Subinterval area

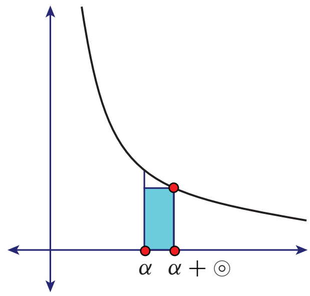

Figure 17 Lower estimate

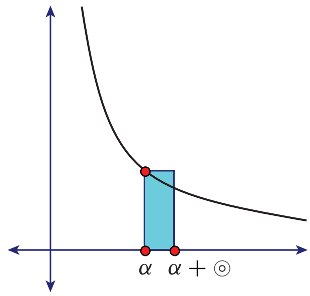

Figure 18 Upper estimate

width of positive infinitesimal length $\circledcirc$, and suppose that $x = \alpha$ is one of the subdivision points. Here, $[\alpha, \alpha + \circledcirc]$ is a typical subinterval of the partition.

Rotating the shaded region trapped between the curve and the x-axis interval $[\alpha, \alpha + \circledcirc]$, as shown in FIGURE 16, about the y-axis, produces a solid best described as a piece of pipe with a curved top edge and a wall thickness of $\circledcirc$. The curved top edge makes this volume a bit more difficult to calculate than we would like. Instead, consider the cylindrical shells that form a lower bound and an upper bound for this pipe. Specifically, the volume of this pipe is bigger than the cylindrical shell of wall thickness $\circledcirc$ produced by using the lower height $\frac{1}{\alpha+\circledcirc}$ at the right endpoint of the interval, as shaded in FIGURE 17, but smaller than the shell of thickness $\circledcirc$ produced by using the higher height $\frac{1}{\alpha}$ of the left endpoint, as shaded in FIGURE 18. The volume of an actual piece of pipe with curved top lies between these bounds; that is,

$$\mathrm{Vol}_{\text{right endpoint shell}} \le \mathrm{Vol}_{\text{pipe}} \le \mathrm{Vol}_{\text{left endpoint shell}} .$$

This yields

$$\frac{\pi \circledcirc (2\alpha + \circledcirc)}{\alpha + \circledcirc} \le \mathrm{Vol}_{\text{pipe}} \le \frac{\pi \circledcirc (2\alpha + \circledcirc)}{\alpha}.$$

Simplification produces

$$2\pi \circledcirc - \frac{\pi \circledcirc^2}{\alpha + \circledcirc} \le \mathrm{Vol}_{\text{pipe}} \le 2\pi \circledcirc + \frac{\pi \circledcirc^2}{\alpha}.$$

Since $\alpha + \circledcirc > \alpha$, we have

$$2\pi \circledcirc - \frac{\pi \circledcirc^2}{\alpha} \le \mathrm{Vol}_{\text{pipe}} \le 2\pi \circledcirc + \frac{\pi \circledcirc^2}{\alpha}. \tag{4}$$

Thus, the volume of a piece of pipe is within $\frac{\pi \circledcirc^2}{\alpha}$ of $2\pi \circledcirc$.

While Torricelli's solution is more straightforward than Roberval's, at this point his method is more difficult to reconcile with infinitesimals because of the complexity added by the infinite discontinuity at the origin. We'll have to split the argument into two pieces of consideration, infinitesimal values of α and non-infinitesimal values of α.

Let's start by considering the volume produced by the partitions with non-infinitesimal values of α. When α is non-infinitesimal, $\frac{\pi \circledcirc^2}{\alpha}$ is infinitesimal with respect to $2\pi \circledcirc$. That is, $2\pi \circledcirc / \left(\frac{\pi \circledcirc^2}{\alpha}\right) = \frac{2\alpha}{\circledcirc}$ is infinite. This allows us to simplify Inequality (4) and say that the volume of each pipe resulting from a non-infinitesimal α lies between $2\pi \circledcirc + \circledcirc'$ and $2\pi \circledcirc - \circledcirc'$, where $\circledcirc'$ is infinitesimal with respect to $\circledcirc$.

Now, we need to sum the volumes from these pipes. We started with $\frac{c}{\circledcirc}$ intervals, but we have excluded all that are infinitely close to 0, that is, those where α is infinitesimal.

Let r be the number of intervals that we have eliminated. Notice that $r\circledcirc$ must be infinitesimal, else it is not infinitely close to 0 and we have excluded a non-infinitesimal α. This gives

$$\sum_{\boxed{\alpha}\neq 0} [2\pi \circledcirc - \circledcirc'] \leq \sum_{\boxed{\alpha}\neq 0} \mathrm{Vol}_{\mathrm{pipe}} \leq \sum_{\boxed{\alpha}\neq 0} [2\pi \circledcirc + \circledcirc'],$$

where $\circledcirc'$ is infinitesimal with respect to $\circledcirc$. Let's consider the two components of the summations on the left and right. Since there are $\frac{c}{\circledcirc} - r$ non-infinitesimal α, the first part of each simplifies to

$$\sum_{\boxed{\alpha}\neq 0} 2\pi\circledcirc = \left(\frac{c}{\circledcirc} - r\right) \cdot 2\pi\circledcirc = 2\pi c - 2\pi \circledcirc r.$$

As noted above, $r\circledcirc$ is infinitesimal. Thus,

$$\boxed{\sum_{\boxed{\alpha}\neq 0} 2\pi\circledcirc} = \boxed{2\pi c} - \boxed{2\pi \circledcirc r} = 2\pi c.$$

For the second contributor to these summations, $\sum \circledcirc'$, we only need to note that $\circledcirc'$ is infinitesimal with respect to $\circledcirc$. Since we are summing over $\frac{c}{\circledcirc} - r$ such α, the sum is at most infinitesimal. Therefore,

$$\boxed{\sum_{\boxed{\alpha}\neq 0} \circledcirc'} = 0,$$

and we have

$$\boxed{\sum_{\boxed{\alpha}\neq 0} \mathrm{Vol}_{\mathrm{pipe}}} = 2\pi c.$$

Now let's consider the volume produced by the partitions where α is infinitesimal. By Inequality (4) we have

$$\begin{aligned}
\sum_{\boxed{\alpha}=0} \mathrm{Vol}_{\mathrm{pipe}} &\leq \sum_{\boxed{\alpha}=0} 2\pi \circledcirc + \frac{\pi\circledcirc^2}{\alpha} \\
&\leq \sum_{\boxed{\alpha}=0} 2\pi \circledcirc + \frac{\pi\circledcirc^2}{\circledcirc}, \quad \text{since each } \alpha \geq \circledcirc, \\
&= \sum_{\boxed{\alpha}=0} 3\pi\circledcirc \\
&= r(3\pi\circledcirc),
\end{aligned}$$

since there are r such infinitesimal α.

Since $r\circledcirc$ is infinitesimal, we have

$$\boxed{\sum_{\boxed{\alpha}=0} \mathrm{Vol}_{\mathrm{pipe}}} \leq \boxed{3\pi r\circledcirc} = 0.$$

Thus, the infinitesimal values of α contribute no real volume to the horn.

In conclusion, since the volume of the horn is a real number, and we have bounded the standard part of the sum of the volumes of these pipes above and below by $2\pi c$, we have

$$\text{Vol}_{\text{horn}} = \boxed{\sum_{\alpha} \text{Vol}_{\text{pipe}}} = 2\pi c.$$

Conclusion

We have been able to salvage some wonderful intuition applied to interesting problems through the use of infinitesimals. If the reader would like to verify further seventeenth-century "proofs" by indivisibles using infinitesimal calculus, we suggest Margaret Baron's *The Origins of the Infinitesimal Calculus* [**1**] as a rich source of such proofs. Baron presents arguments like Roberval's by Fermat, Descartes and Christiaan Huygens (1629–1695), each of whom determined the area under a cycloid using indivisibles. John Wallis used indivisibles to determine the volume of a cylindrical section on a semicircular base. Luca Valerio (1552–1618), who inspired Cavalieri, found the volume of a hemisphere in a manner that blended indivisibles with the modern concept of limit.

Finally, Baron details many results via indivisibles produced by Kepler, who used the method with abandon. Of him, Baron writes that he "sought to demonstrate in the simplest possible way the existence of mathematical form and structure in the external world and, upheld by his Platonic-Pythagorean philosophy, he often allowed faith, analogy and intuition to guide him when traditional methods failed" [**1**]. With Robinson's theory of infinitesimals, the intuition of these prolific mathematicians who expanded the boundaries of seventeenth century mathematics rests on a solid foundation.

REFERENCES

1. M. Baron, *The Origins of the Infinitesimal Calculus*, Dover, New York, 1969.
2. C. Boyer, *The History of the Calculus and Its Conceptual Development*, Dover, New York, 1949.
3. C. H. Edwards, Jr., *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979. http://dx.doi.org/10.1007/978-1-4612-6230-5
4. R. Goldblatt, *Lectures on the Hyperreals*, Springer-Verlag, New York, 1998. http://dx.doi.org/10.1007/978-1-4612-0615-6
5. J. M. Henle and E. M. Kleinberg, *Infinitesimal Calculus*, Dover, New York, 2003.
6. R. F. Hoskins, *Standard and Nonstandard Analysis*, Ellis Horwood, London, 1990.
7. H. J. Keisler, *Elementary Calculus: An Infinitesimal Approach*, 2nd ed., Prindle, Weber & Schmidt, Boston, 1986. Currently available online at http://www.math.wisc.edu/~keisler/calc.html.
8. P. Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, Oxford, New York, 1996.
9. A. Robinson, *Non-standard Analysis*, North-Holland, Amsterdam, 1966.
10. J. Stedall, *The Arithmetic of Infinitesimals: John Wallis 1656*, Springer, New York, 2004.
11. D. Struik, *A Source Book in Mathematics, 1200–1800*, Princeton University Press, Princeton, 1986.

Summary In this article, we describe clever arguments by Torricelli and Roberval that employ indivisibles to find the volume of Gabriel's trumpet and the area under the cycloid. We detail 17th-century objections to these non-rigorous but highly intuitive techniques, as well as the controversy surrounding indivisibles. After reviewing the fundamentals of infinitesimal calculus and its rigorous footing provided by Robinson in the 1960s, we are able to revisit the 17th-century solutions. In changing from indivisible to infinitesimal-based arguments, we salvage the beautiful intuition found in these works.

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NOTES

Lebesgue's Road to Antiderivatives

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The following theorem is well known:

THEOREM. *If* $f : [a, b] \to \mathbb{R}$ *is continuous, then* f *has antiderivatives.*

In most calculus courses and books, this is proved as part of the fundamental theorem of calculus, which states that, for a fixed $x_0 \in (a, b)$ and every x where f is continuous, the integral function $F(x) = \int_{x_0}^{x} f(t)\,dt$ is differentiable and satisfies $F'(x) = f(x)$. Since the statement of the theorem does not involve an integral, it is natural to ask whether the theorem can be proved without using the theory of definite integrals. In fact, Henri Lebesgue himself provided such a proof of this theorem in 1904–1905 [**1**, pp. 85–89; **2**]. In this paper, we revisit Lebesgue's proof.

Lebesgue's method is based on piecewise linear approximations of continuous functions. The existence of antiderivatives for these special types of functions is straightforward, and we can show that their limit also has antiderivatives. Remarkably, this approach also yields a proof of a mean-value-type inequality and the uniform convergence of antiderivatives of uniformly convergent sequences of continuous functions.

Both approaches rely on a theorem of E. Heine [**4**, pp. 143, 263], which states that continuous functions defined on bounded closed intervals are uniformly continuous. In the traditional approach, it is used to show that continuous functions are integrable on bounded closed intervals. In Lebesgue's approach, it provides piecewise linear approximations of continuous functions. However, this latter method, contrary to the traditional one, does not require the notion of supremum or infimum and oscillation. As Lebesgue wrote in [**2**]:

> "If one is limited to continuous functions in the whole course, then it [the traditional method] can be replaced by the following slightly different method which seems simpler for me."

In this note we recall Lebesgue's ideas of [**2**] in a somewhat modernized form (but keeping some of the original notation) and fill in some details according to his closing remarks, suggesting some modification of the arguments. Although Lebesgue did use the Mean Value Theorem in his proof (and also the notion of supremum, infimum, and oscillation), he noted that he did it only to shorten the presentation and it can be avoided easily. We shall accept Lebesgue's challenge and avoid using the Mean Value Theorem.

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Lebesgue's proof revisited

We assume, as Lebesgue did, that f is continuous on the closed interval $[a, b]$.

We first show the existence of antiderivatives for the class of piecewise linear functions. The general case will follow by approximations.

Piecewise linear functions have antiderivatives A linear function $f(x) = mx + n$ has an antiderivative $(m/2)x^2 + nx + K$, where K is any constant. Note that these are the only quadratic antiderivatives, since the coefficients of x^2 and x are uniquely determined by the derivative. (In fact, these are the only antiderivatives, but we do not need this result.)

Suppose now that f is piecewise linear, which means that $f(x) = m_i x + n_i$ in $[a_i, a_{i+1}]$ for $i = 0, \dots, p-1$ where $a = a_0 < a_1 < a_2 < \cdots < a_p = b$, and the following matching conditions hold (see FIGURE 1):

$$m_i a_{i+1} + n_i = m_{i+1} a_{i+1} + n_{i+1} \quad (i = 0, \dots, p-2).$$

For a piecewise linear function f, a piecewise quadratic antiderivative F can be constructed as follows. Let

$$F(x) = \frac{m_0}{2} x^2 + n_0 x - \frac{m_0}{2} a_0^2 - n_0 a_0 \quad (x \in [a_0, a_1])$$

and define (recursively) for $i = 1, \dots, p-1$,

$$F(x) = \frac{m_i}{2} x^2 + n_i x + F(a_i) - \frac{m_i}{2} a_i^2 - n_i a_i \quad (x \in (a_i, a_{i+1}]).$$

The constants $F(a_i) - (m_i/2)a_i^2 - n_i a_i$ are chosen so that F is continuous at $x = a_1, a_2, \dots, a_{p-1}$. The matching conditions ensure the differentiability of F at these points. It follows from the above construction that F is differentiable in (a, b) and $F' = f$. Further, $F(a) = 0$ also holds.

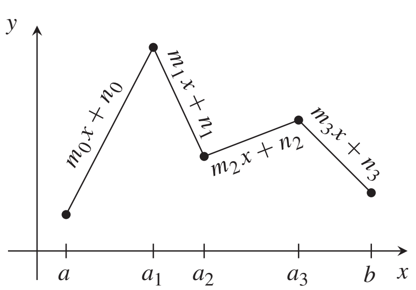

Figure 1 Piecewise linear function

It is also worth noticing that F is the unique piecewise quadratic antiderivative of the piecewise linear function f such that $F(a) = 0$, because the coefficients of the quadratic polynomials are uniquely determined by the coefficients of the derivative, the matching conditions, and the requirement that $F(a) = 0$. Consequently, a piecewise linear function admits a unique piecewise quadratic antiderivative up to an additive constant.

An inequality We now prove a simple inequality for piecewise quadratic functions. This will turn out to be a key result in the proof of the theorem.

LEMMA. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a continuous piecewise linear function. Assume that* $F\colon [a,b] \to \mathbb{R}$ *is a continuous piecewise quadratic function such that* F *is differentiable in* (a,b) *and* $F' = f$. *Then*

$$(y-x)\ \min_{[x,y]} f \ \le\ F(y)-F(x) \ \le\ (y-x)\ \max_{[x,y]} f \tag{1}$$

for all $x, y \in [a,b]$ *with* $x < y$.

Proof. We may assume that f and F have the same form as above. First, suppose that x, y are in the same subinterval, $x, y \in [a_i, a_{i+1}]$, with $x < y$. Then

$$F(y)-F(x) = (y-x)\left(m_i\frac{y+x}{2} + n_i\right) = (y-x) f\left(\frac{y+x}{2}\right). \tag{2}$$

Since $(y+x)/2 \in [x,y]$, we conclude that $f(\frac{y+x}{2})$ is between $\min_{[x,y]} f$ and $\max_{[x,y]} f$. So (1) holds on a subinterval $[a_i, a_{i+1}]$.

In the general case, if $x \in [a_i, a_{i+1}]$, $x < y \in [a_j, a_{j+1}]$, then we may expand $F(y) - F(x)$ as a telescoping sum

$$F(y)-F(x) = (F(y)-F(a_j)) + (F(a_j)-F(a_{j-1})) + \cdots + (F(a_{i+1})-F(x)).$$

By applying (1) on each subinterval, after summation (1) follows on the whole $[a,b]$. We note that the existence of the maximum and minimum in (1) does not require the Extreme Value Theorem. It is a consequence of the fact that f is continuous and piecewise monotone, so that the extrema are attained at the endpoints of the interval or at some point where f changes monotonicity. ∎

Continuous functions Now that we have proved the theorem for a continuous piecewise linear function f, we next consider an arbitrary continuous function f. In this part of the proof of the theorem, inequalities (1) will play a central role.

Let now $f\colon [a,b] \to \mathbb{R}$ be an arbitrary continuous function. We show that f has antiderivatives. Since f is uniformly continuous by Heine's theorem, for every positive integer n there is $\delta_n > 0$, such that $|f(x) - f(y)| < 1/n$ whenever $|x - y| < \delta_n$. Take an arbitrary partition $a = a_0 < a_1 < a_2 < \cdots < a_r = b$ of the interval $[a,b]$ such that $a_{i+1} - a_i < \delta_n$ $(i = 0, \ldots, r-1)$, and define the piecewise linear function f_n to be equal to f at a_i and to be linear on $[a_i, a_{i+1}]$ for $i = 0, \ldots, r-1$ (see FIGURE 2).

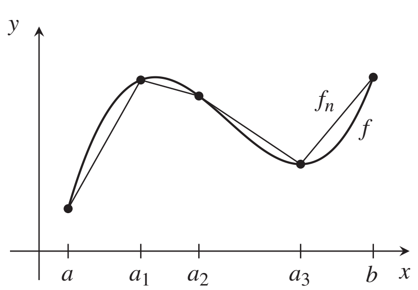

Figure 2 Piecewise linear approximation

If $x \in [a_i, a_{i+1}]$, then by the linearity, $f_n(x)$ is between the maximum and minimum of the values $f(a_i)$, $f(a_{i+1})$, and $f(x)$; hence, $f_n(x)$ and $f(x)$ differ by less than $1/n$.

So

$$|f_n(x) - f(x)| < 1/n \quad (x \in [a, b]). \tag{3}$$

Now consider the function $f_n - f_m$ for positive integers n, m. First note that by (3),

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &< \frac{1}{n} + \frac{1}{m} \quad (x \in [a, b]). \end{aligned} \tag{4}$$

In addition, the piecewise linear functions f_n and f_m have piecewise quadratic antiderivatives F_n and F_m, respectively, with the property $F_n(a) = F_m(a) = 0$. Then $F_n - F_m$ is an antiderivative of $f_n - f_m$ and $F_n(a) - F_m(a) = 0$. Observe that the function $f_n - f_m$ is also piecewise linear, since the partitions corresponding to f_n and f_m divide $[a, b]$ into subintervals on which both f_n and f_m are linear, and so is their difference. Likewise, $F_n - F_m$ is piecewise quadratic. Therefore, we may apply (1) to the piecewise linear function $f_n - f_m$ and its piecewise quadratic antiderivative $F_n - F_m$, and we obtain for $x \in [a, b]$

$$|(F_n(x) - F_m(x)) - (F_n(a) - F_m(a))| \le |x - a| \max_{[a,x]} |f_n - f_m|.$$

Thus, $F_n(a) = F_m(a) = 0$ (in fact, only $F_n(a) = F_m(a)$ is necessary) and (4) imply

$$|F_n(x) - F_m(x)| \le |b - a| \left(\frac{1}{n} + \frac{1}{m} \right) \quad (x \in [a, b]). \tag{5}$$

This means that $(F_n(x))$ is a Cauchy sequence for every $x \in [a, b]$. Therefore, it is convergent, and $F_n(x) \to F(x)$ for some $F\colon [a, b] \to \mathbb{R}$.

We prove that F is differentiable in (a, b) and $F' = f$; moreover, $F'(a) = f(a)$ and $F'(b) = f(b)$ as one-sided derivatives. To this end, fix $x_0 \in [a, b]$ and let $\varepsilon > 0$. Since f is continuous at x_0, there is $\delta > 0$ such that

$$|f(x_0) - f(x)| < \varepsilon \quad \text{for} \quad |x - x_0| < \delta, x \in [a, b]. \tag{6}$$

We show that if $0 < |x_0 - x| < \delta$, $x \in [a, b]$, then

$$f(x_0) - \varepsilon \le \frac{F(x) - F(x_0)}{x - x_0} \le f(x_0) + \varepsilon, \tag{7}$$

which implies that $F'(x_0) = f(x_0)$. First, for $x \in [a, b]$, $x \ne x_0$, we have by (1) that

$$\min_{[x_0,x]} f_n \le \frac{F_n(x) - F_n(x_0)}{x - x_0} \le \max_{[x_0,x]} f_n, \tag{8}$$

where for $x < x_0$ the maximum and minimum are taken over $[x, x_0]$. Let $x_n^{\max} \in [x_0, x]$ be some point where $\max_{[x_0,x]} f_n$ is attained. Then, for $0 < |x - x_0| < \delta$, the inequalities (3) and (6) imply

$$\max_{[x_0,x]} f_n = f_n(x_n^{\max}) < f(x_n^{\max}) + \frac{1}{n} < f(x_0) + \varepsilon + \frac{1}{n}.$$

Analogously, we have for $0 < |x - x_0| < \delta$,

$$f(x_0) - \frac{1}{n} - \varepsilon < \min_{[x_0,x]} f_n.$$

So by (8) we obtain

$$f(x_0) - \frac{1}{n} - \varepsilon < \frac{F_n(x) - F_n(x_0)}{x - x_0} < f(x_0) + \frac{1}{n} + \varepsilon.$$

Now, (7) follows as $n \to \infty$.

There is more beneath

The Mean Value Theorem A key step in the proof was the lemma, along with inequality (1). This inequality would follow from the *Mean Value Theorem*, which says that if F is a continuous function on the closed interval $[a, b]$ and differentiable in (a, b), then

$$F(b) - F(a) = F'(\xi)(b - a) \quad \text{for some } \xi \in (a, b).$$

We did not rely on the Mean Value Theorem. Instead, we can deduce a variant of it from the lemma. Indeed, by (3) and (8),

$$\min_{[x_0,x]} f - \frac{1}{n} \le \min_{[x_0,x]} f_n \le \frac{F_n(x) - F_n(x_0)}{x - x_0} \le \max_{[x_0,x]} f_n \le \max_{[x_0,x]} f + \frac{1}{n}.$$

So as $n \to \infty$ it follows that

$$\min_{[x_0,x]} f \le \frac{F(x) - F(x_0)}{x - x_0} \le \max_{[x_0,x]} f. \tag{9}$$

We obtained that if $f\colon [a, b] \to \mathbb{R}$ is continuous, then it admits an antiderivative $F\colon [a, b] \to \mathbb{R}$ such that the mean value inequality (9) holds (where the existence of extrema follows from the Extreme Value Theorem). It is not difficult to verify in an elementary way, without using the Mean Value Theorem, that antiderivatives are unique up to an additive constant (see [**3**] for some references). Thus, inequality (9) holds for every antiderivative of f.

As we mentioned before, Lebesgue did use the Mean Value Theorem in his proof, but only to shorten the presentation. According to him, though the Mean Value Theorem is a simple and important result, its rigorous proof is rarely understood and in some courses it may be advantageous to replace it with a result like (9). Some decades later, the role of the Mean Value Theorem in the calculus curriculum was also intensively discussed; see [**3**] for a list of references.

Area and antiderivatives The equality (2) can be expressed graphically. Suppose that f is a positive continuous piecewise linear function, and choose $x = a_i$, $y = a_{i+1}$ in (2). Then

$$F(a_{i+1}) - F(a_i) = (a_{i+1} - a_i) f\left(\frac{a_i + a_{i+1}}{2}\right),$$

which means that the area of the trapezoid under the graph of the piecewise linear function f on the interval $[a_i, a_{i+1}]$ equals $F(a_{i+1}) - F(a_i)$ (see FIGURE 3). By summing the areas of the trapezoids from $i = 0$ to $i = p - 1$, we obtain that the area under the graph of the piecewise linear function f is $F(b) - F(a)$, where F is a continuous piecewise quadratic antiderivative of f. Since antiderivatives are unique up to an additive constant, it holds for every antiderivative of f.

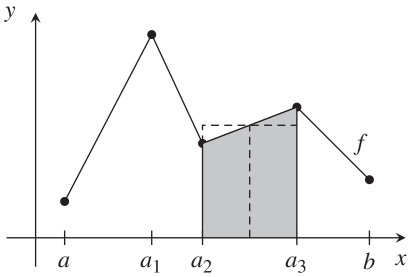

Figure 3 Area under a piecewise linear function

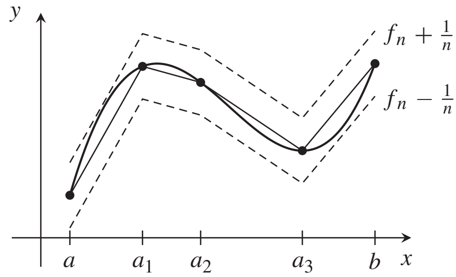

Figure 4 Area under a continuous function

We can extend this result to an arbitrary positive continuous function $f\colon [a,b] \to \mathbb{R}$. Choose a piecewise linear approximation f_n to f with the property (3). Then f is between the functions $f_n - \frac{1}{n}$ and $f_n + \frac{1}{n}$. If n is large enough, then $f_n - \frac{1}{n}$ is also positive; thus the area under the graph of f is between the area under $f_n - \frac{1}{n}$ and the area under $f_n + \frac{1}{n}$ (see FIGURE 4). Since $F_n(x) - \frac{x}{n}$ and $F_n(x) + \frac{x}{n}$ are antiderivatives of $f_n - \frac{1}{n}$ and $f_n + \frac{1}{n}$, respectively, it follows that

$$F_n(b) - F_n(a) - \frac{b-a}{n} \le \text{area under the graph of } f \le F_n(b) - F_n(a) + \frac{b-a}{n}.$$

Now, as $n \to \infty$ we obtain that

$$\text{area under the graph of } f = F(b) - F(a),$$

where $F\colon [a,b] \to \mathbb{R}$ is continuous and $F' = f$ in (a,b). This can be regarded as an "integral-free" version of the so-called second fundamental theorem of calculus. It can also be seen as a version of the Newton–Leibniz formula for a (positive) continuous function $f\colon [a,b] \to \mathbb{R}$, which states that $\int_a^b f(t)\,dt = F(b) - F(a)$ [**4**, p. 286]. In other words, we have proved the Riemann integrability of continuous functions, and the approach is different from the standard one.

Uniform convergence and antiderivatives Observe that inequalities (3) and (5) express the uniform convergence of (f_n) and (F_n), respectively. Consequently, as Lebesgue did in [**1**, p. 88], we proved much more.

THEOREM. *Let $f_n\colon [a,b] \to \mathbb{R}$ $(n = 1, 2\ldots)$ be continuous functions with continuous antiderivatives $F_n\colon [a,b] \to \mathbb{R}$, such that $F_n(a) = F_1(a)$ $(n = 1, 2\ldots)$. If $f_n \to f$ uniformly in $[a,b]$, then $F_n \to F$ uniformly in $[a,b]$, where $F\colon [a,b] \to \mathbb{R}$ is continuous and $F' = f$ in (a,b).*

REFERENCES

1. H. Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, Gauthier-Villars, Paris, 1904.
2. ———, Remarques sur la définition de l'intégrale, *Bull. Sci. Math.* **29** (1905) 272–275.
3. R. S. Smith, Rolle over Lagrange—Another Shot at the Mean Value Theorem, *College Math. J.* **17** (1986) 403–406.
4. M. Spivak, *Calculus*, 3rd ed., Publish or Perish, Berkeley, CA, 1994.

Summary The traditional way of proving the existence of antiderivatives of continuous functions is through the concept of definite integrals. In the years 1904–1905, H. Lebesgue provided an alternative proof of this result not relying on the theory of integrals. His method is based on piecewise linear approximations of continuous functions, which also yields the mean value inequality as a by-product. In this note we recall Lebesgue's ideas.

Unified Proofs of the Error Estimates for the Midpoint, Trapezoidal, and Simpson's Rules

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Cruz-Uribe and Neugebauer, in this MAGAZINE [**1**], gave an elementary proof of the error estimate for the trapezoidal rule using what was referred to as integration by parts "backwards." Hai and Smith [**2**] then used the same method to give an elementary proof of the error estimate for Simpson's rule. Fazekas and Mercer [**3**] extended these ideas to prove error estimates for both the midpoint rule and Simpson's rule.

The purpose of this article is to present another technique for proving the error estimates, one that is also unified and elementary, and more direct.

Let g be a real-valued function on the fixed interval $[c, d]$. For the midpoint and trapezoidal rules, we assume that $g''(t)$ exists and satisfies $|g''(t)| \leq M$ for all t in (c, d), for some positive bound M. In the case of Simpson's rule, we assume that $g^{(iv)}(t)$ exists and satisfies $|g^{(iv)}(t)| \leq M$ for all $t \in (c, d)$. We will show that these error bounds hold.

MIDPOINT RULE ERROR.

$$\left|\int_c^d g(t)dt - (d-c)g\left(\frac{c+d}{2}\right)\right| \leq \frac{M\,(d-c)^3}{24}.$$

TRAPEZOIDAL RULE ERROR.

$$\left|\int_c^d g(t)dt - (d-c)\left(\frac{g(c)+g(d)}{2}\right)\right| \leq \frac{M\,(d-c)^3}{12}.$$

SIMPSON'S RULE ERROR.

$$\left|\int_c^d g(t)dt - \frac{(d-c)}{6}\left(g(c) + 4g\left(\frac{c+d}{2}\right) + g(d)\right)\right| \leq \frac{M\,(d-c)^5}{2880}.$$

To this end, we will determine how the error varies with the length of the interval. Fix $a = (c+d)/2$, the midpoint of the interval, and let h be a variable satisfying $0 \leq h \leq (d-c)/2$, so that $[a-h, a+h] \subseteq [c, d]$.

Define

$$E(h) = \int_{a-h}^{a+h} g(t)dt - h\left(Ag(a-h) + Bg(a) + Cg(a+h)\right),$$

where

$$(A, B, C) = \begin{cases} (0, 2, 0) & \text{for the midpoint rule,} \\ (1, 0, 1) & \text{for the trapezoidal rule, and} \\ (\frac{1}{3}, \frac{4}{3}, \frac{1}{3}) & \text{for Simpson's rule.} \end{cases}$$

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Differentiating $E(h)$ with respect to h, we have

$$\begin{aligned} E'(h) &= g(a+h) + g(a-h) - \big(Ag(a-h) + Bg(a) + Cg(a+h)\big) \\ &\quad - h\left(-Ag'(a-h) + Cg'(a+h)\right) \\ &= (1-C)g(a+h) + (1-A)g(a-h) - Bg(a) \\ &\quad + h\left(Ag'(a-h) - Cg'(a+h)\right), \text{ and} \\ E''(h) &= g'(a+h) - g'(a-h) - 2\left(-Ag'(a-h) + Cg'(a+h)\right) \\ &\quad - h\left(Ag''(a-h) + Cg''(a+h)\right). \end{aligned} \tag{1}$$

Evaluating at $h = 0$ gives $E(0) = 0$ and

$$\begin{aligned} E'(0) &= (2 - (A+B+C))g(a), \\ E''(0) &= (2(A-C))g'(a). \end{aligned}$$

Since $A + B + C = 2$ and $A - C = 0$ for each of the rules we are considering, in each of these cases we have $E(0) = E'(0) = E''(0) = 0$.

Midpoint rule

For the midpoint rule, $(A, B, C) = (0, 2, 0)$, so (1) gives

$$E''(h) = g'(a+h) - g'(a-h).$$

By the Mean Value Theorem, we have

$$E''(h) = g'(a+h) - g'(a-h) = g''(\beta)2h$$

for some β in $(a-h, a+h)$. Therefore, $-2Mh \le E''(h) \le 2Mh$. Integrating over $[0, h]$ gives

$$-Mh^2 \le E'(h) - E'(0) \le Mh^2,$$

hence $-Mh^2 \le E'(h) \le Mh^2$. Another such integration gives

$$\frac{-Mh^3}{3} \le E(h) - E(0) \le \frac{Mh^3}{3}.$$

When $h = (d-c)/2$, we have

$$\left|E\left(\frac{d-c}{2}\right)\right| = \left|\int_c^d g(t)dt - (d-c)g\left(\frac{c+d}{2}\right)\right| \le M\frac{(d-c)^3}{24}.$$

This is the midpoint rule error estimate.

Equality holds for $g(t) = t^2$ on $[0, 2]$, with $g''(t) \equiv 2 = M$.

Trapezoidal rule

For the trapezoidal rule, $(A, B, C) = (1, 0, 1)$, so (1) gives

$$E''(h) = -\left(g'(a+h) - g'(a-h)\right) - \left(g''(a-h) + g''(a_h)\right)h.$$

By the Mean Value Theorem, we have

$$E''(h) = -g''(\beta)2h - \big(g''(a-h) + g''(a+h)\big)\, h$$

for some β in $(a-h, a+h)$. Therefore, $-4Mh \le E''(h) \le 4Mh$. Integrating twice as before yields

$$\frac{-4Mh^3}{2\cdot 3} \le E(h) \le \frac{4Mh^3}{2\cdot 3}.$$

When $h = (d-c)/2$, we have

$$\begin{aligned}\left|E\left(\frac{d-c}{2}\right)\right| &= \left|\int_c^d g(t)dt - (d-c)\left(\frac{g(c)+g(d)}{2}\right)\right| \\ &\le \frac{2M(c-d)^3}{3\cdot 8} = \frac{M(c-d)^3}{12}.\end{aligned}$$

This is the trapezoidal rule error estimate.

Equality holds for $g(t) = t^2$ on [0, 2], with $g''(t) \equiv 2 = M$.

Simpson's rule

For Simpson's rule, $(A, B, C) = (\frac{1}{3}, \frac{4}{3}, \frac{1}{3})$, so (1) gives

$$E''(h) = \frac{1}{3}\big(g''(a+h) - g''(a-h)\big) - \frac{h}{3}\big(g''(a+h) + g''(a-h)\big).$$

Differentiating,

$$\begin{aligned}E'''(h) &= \frac{1}{3}\big(g''(a+h) + g''(a-h)\big) - \frac{1}{3}\big(g''(a+h) + g''(a-h)\big) \\ &\quad - \frac{h}{3}\big(g'''(a+h) - g'''(a-h)\big) \\ &= -\frac{h}{3}\big(g'''(a+h) - g'''(a-h)\big).\end{aligned}$$

Now, by the Mean Value Theorem,

$$E'''(h) = -\frac{h}{3}\big(g'''(a+h) - g'''(a-h)\big) = -\frac{h}{3} g^{(iv)}(\beta)(2h)$$

for some β in $(a-h, a+h)$. Therefore, $-2Mh^2/3 \le E'''(h) \le 2Mh^2/3$. Three integrations as before yields

$$|E(h)| \le \frac{2Mh^5}{3\cdot 3\cdot 4\cdot 5} = \frac{Mh^5}{90}.$$

When $h = (c+d)/2$, we have

$$\begin{aligned}\left|E\left(\frac{d-c}{2}\right)\right| &= \left|\int_c^d g(t)dt - \frac{(d-c)}{6}\left(g(c) + 4g\left(\frac{c+d}{2}\right) + g(d)\right)\right| \\ &\le \frac{M(d-c)^5}{90\cdot 2^5} = \frac{M(d-c)^5}{2880}.\end{aligned}$$

This is the Simpson's rule error estimate.

Equality holds for $g(t) = t^4$ on [0, 2], with $g^{(iv)}(t) \equiv 4! = 24 = M$.

REFERENCES

1. D. Cruz-Uribe and C. J. Neugebauer, An elementary proof of error estimates for the trapezoidal rule, *Math. Mag.* **76** (2003) 303–3006. http://dx.doi.org/10.2307/3219088
2. D. D. Hai and R. C. Smith, An elementary proof of the error estimates in Simpson's rule, *Math. Mag.* **81** (2008) 295–300.
3. Edward C. Fazekas, Jr. and Peter R. Mercer, Elementary proofs of error estimates for the midpoint and Simpson's rules, *Math. Mag.* **82** (2009) 365–370. http://dx.doi.org/10.4169/002557009X478418

Summary Unified, elementary proofs are given for the error estimates associated with the midpoint rule, the trapezoidal rule, and Simpson's rule.

Alcuin of York

The cover image is a drawing by Amy Uyeki of Arcata, California, based on an illustration published by Andre Thevet in 1584 [**3**].

Alcuin was educated at the cathedral school in York. In 781 he joined Charlemagne's court. He was head of the palace school in Aachen from 781 till 796, and Abbot of Tours until his death in 804.

Alcuin never wrote about Alcuin's sequence, or about the connection of Alcuin's sequence to integer-sided triangles. But he is the presumed author of a remarkable problem book, *Propositiones ad acuendos juvenes,* which appeared in about 800 CE. It is translated by Hadley and Singmaster as "Problems to Sharpen Youths" [**1**], and other versions can be found on the web.

Alcuin's Problem 12 calls for three brothers to share an estate consisting of 30 flasks—10 full of oil, 10 half full, and 10 empty. The brothers are to share the flasks and the oil equally. Alcuin gives only one solution. But there are actually five solutions, as pointed out by David Singmaster in the *College Math. Journal* [**2**].

It turns out that, if $t(n)$ is the number of incongruent triangles with integer sides and perimeter n, then $t(n+3)$ is the number of solutions to Alcuin's problem with n full flasks, n half-full flasks, and n empty flasks. The sequence $t(n)$ is now called "Alcuin's sequence," and it is the subject of the article by Beauregard and Dobrushkin in this issue (p. 280). That article cites further references.

Alcuin's greater contributions to mathematics may have been his promotion of schools, generally. He also promoted the new "Carolingian miniscule" script, which made it easier to copy manuscripts, and so made them more easily available everywhere. If your ancestor's school library was able to afford its own copy of *the Elements,* it may have been due to the influence of Alcuin.

REFERENCES

1. John Hadley and David Singmaster, Problems to sharpen the young, *Math. Gazette* **76**, #475 (March, 1992) 102–126. http://dx.doi.org/10.2307/3620384
2. David Singmaster, Triangles with integer sides and sharing barrels, *College Math. J.* **21** (1990) 278–285.
3. Andre Thevet, *Les Vrais Pourtraits Et Vies Des Hommes Illustres,* 1584. http://openlibrary.org/works/OL1331962W/Les_vrais_pourtraits_et_vies_des_hommes_illustres_grecz_latins_et_payens

Rational Ratios and Concurrent Cevians

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In a triangle, a *cevian* is a line segment with one endpoint at a vertex of the triangle and the other endpoint on the opposite side. Sometimes cevians come in sets of three, like the three medians of a triangle, the three angle bisectors, or the three altitudes. In each of these cases, the three cevians are concurrent: The medians meet at the centroid, the angle bisectors meet at incenter, and the altitudes meet at the orthocenter.

In his 1678 work *De lineis rectis se invicem secantibus statica constructio*, the Italian mathematician Giovanni Ceva (1647–1734) presented the following result (FIGURE 1). We are using the same notation to represent both line segments and their lengths.

THEOREM. *Let ABC be a triangle and let D, E, and F be points on the sides BC, AC, and AB, respectively. Then the lines AD, BE, and CF are concurrent if and only if*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

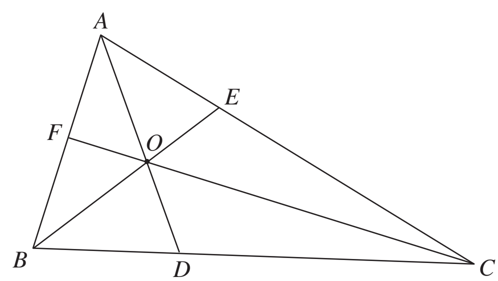

Figure 1 Concurrent cevians

Although Ceva's publication motivates the name "cevian," this result was previously proved by Yusuf al-Mu'taman ibn Hūd, an eleventh-century king of Zaragoza (now in Spain). It is not known whether it was his original work [**4**]. The concept of cevian has received much attention over the years [**2, 3, 5, 6, 7**].

Euler also paid attention to cevians. In an 1812 work titled *Geometrica et spherica quaedam*, he proves the following theorem (with the notation of FIGURE 1).

THEOREM. *Let ABC be a triangle and let D, E, and F be points on the sides BC, AC, and AB, respectively. Then, AD, BE, and CF are concurrent at a point O if and only if the following relation holds:*

$$\frac{AO}{OD} \cdot \frac{BO}{OE} \cdot \frac{CO}{OF} = \frac{AO}{OD} + \frac{BO}{OE} + \frac{CO}{OF} + 2.$$

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Atzema and White [**1**] discuss this work by Euler and present many extensions and applications.

We can put

$$\frac{AO}{OD} = x, \quad \frac{BO}{OE} = y, \quad \frac{CO}{OF} = z.$$

With this notation, Euler's theorem states that the three cevians are concurrent at O if and only if

$$xyz = x + y + z + 2. \tag{1}$$

The values x, y, and z are precisely the ratios in which the cevians are divided by their common point O. We have pointed out that medians are a special case of concurrent cevians. It is well known that the centroid of a triangle divides each median in the ratio 2 : 1. This corresponds to the fact that (2, 2, 2) is a solution of equation (1).

The question naturally arises of whether there are other sets of concurrent cevians, different from the medians, such that their common point divides them in three integral (or at least rational) ratios. The main goal of this paper is to answer this question. We will find all integer and all positive rational solutions of (1) and we will see how to construct a set of concurrent cevians leading to any particular positive rational solution.

The positive integer solutions of (1)

Equation (1) is invariant under permutations of (x, y, z).

We can rewrite it as

$$z(xy - 1) = x + y + 2. \tag{2}$$

This is equivalent to (1), whatever the signs of x, y, z. When x, y, z are all positive, it implies that $xy - 1$ is a divisor of $x + y + 2$ and hence $x + y + 2 \geq xy - 1$. This inequality can be written as $(x - 1)(y - 1) \leq 4$ with $x, y \geq 1$. This implies that $x \in \{1, 2, 3, 4, 5\}$, and by direct inspection we get the following solutions, up to permutation:

$$(x, y, z) = (2, 2, 2),\ (1, 2, 5),\ \text{or } (1, 3, 3).$$

If we are interested in all integer (not only positive) solutions, we have the following result.

PROPOSITION. *The equation $xyz = x + y + z + 2$ has only the following integer solutions, up to permutation:*

$$(x, y, z) \in \{(2, 2, 2), (1, 2, 5), (1, 3, 3), (0, \alpha, -2 - \alpha), (-1, -1, \beta)\}$$

with $\alpha, \beta \in \mathbb{Z}$.

Proof. We have already studied the case $x, y, z > 0$. Let us turn to the other possibilities.

First, we examine the case when at least one of the three variables, say x, is 0. Thus, we get the equation $0 = y + z + 2$ and hence we obtain the solutions, up to permutation:

$$(x, y, z) = (0, \alpha, -2 - \alpha).$$

Now, we assume that $xyz \neq 0$ and at least one of them is negative. Three cases arise:

- If $x, y, z < 0$, put $x' = -x$, $y' = -y$, and $z' = -z$, so that equation (2) becomes $z'(x'y' - 1) = x' + y' - 2$. It follows that $x'y' - 1 \leq x' + y' - 2$; i.e., $(x' - 1)(y' - 1) \leq 0$. Since $x', y' > 0$ we get, up to permutation, $(x', y', z') = (1, 1, \beta)$ with $1 \leq \beta \in \mathbb{Z}$. Finally, these solutions lead to the following solutions of equation (1) up to permutation:

$$(x, y, z) = (-1, -1, -\beta), \ 1 \leq \beta \in \mathbb{Z}.$$

- If $x, y < 0$ and $z > 0$, we put $x' = -x$, $y' = -y$ and equation (2) becomes $x'y'z = -x' - y' + z + 2$. This can be written as $z(x'y' - 1) = -x' - y' + 2$ and since $0 \leq z(x'y' - 1) = -x' - y' + 2 \leq 0$, it follows that $x' + y' = 2$, so the only positive solutions are $x' = y' = 1$. Hence, the solutions of our original equation in this case are, up to permutation:

$$(x, y, z) = (-1, -1, \beta), \ 1 \leq \beta \in \mathbb{Z}.$$

- If $x < 0$ and $y, z > 0$, we put $x' = -x$ and equation (2) becomes $-x'yz = -x' + y + z + 2$. Again, we get that $0 \geq x'(1 - yz) = y + z + 2 \geq 0$. Consequently, $y + z = -2$, which is impossible and this case has no solution. ■

Geometric interpretations

We have already seen that the solution (2, 2, 2) corresponds to the case when the considered cevians are the medians of the triangle ABC. In this case, the common point O is the incenter and it divides the three medians in the ratio 2 : 1.

Also observe that the cases involving zero or negative integers do not admit any interesting geometric realizations:

(i) If $x = 0$, it follows that $A = O = F = E$ and the situation is of no interest.

(ii) Solutions involving negative integers could be acceptable if O were to lie outside the triangle ABC and we consider oriented segments. Nevertheless, we have seen that in this case at least one of the variables is -1. But the ratio $AO/OD = x = -1$ cannot occur for a point O in the plane. (We could consider O to be the point at infinity, and relate it to the case of parallel cevians.)

Now we will focus on the other two positive cases.

The solution (1, 3, 3) Let ABC be a triangle (see FIGURE 2) and let AD be the median passing through A. Let O be the midpoint of AD and consider the cevians BO and CO, which determine the points E and F, respectively. We show that

$$\frac{BO}{OE} = \frac{CO}{OF} = 3.$$

Assume, without loss of generality, that the area of ABC is 1. Then it is clear that the area of BOC is $1/2$ so that the area of DOC is $1/4$. Since the area of ADC is also $1/2$, it follows that the area of AOC is $1/4$. Hence, $OE = BE/4$. The same argument leads to $OF = CF/4$ and we are done.

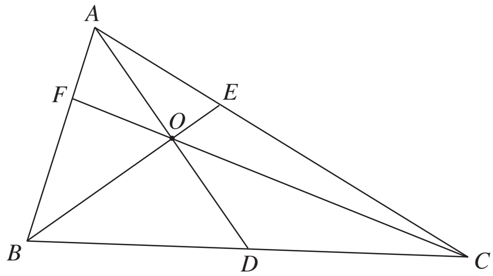

Figure 2 The case $(1, 3, 3)$

The solution (1, 2, 5) This case admits a similar approach. Let ABC be a triangle (see FIGURE 3) and let D be a point in BC such that $BD = 2DC$. We consider the cevian AD (we could call it a *tertian*) and let O be the midpoint of AD. Now, the cevians BO and CO determine the points E and F, respectively. Reasoning as in the previous case, we can prove that

$$\frac{BO}{OE} = 5, \quad \frac{CO}{OF} = 2.$$

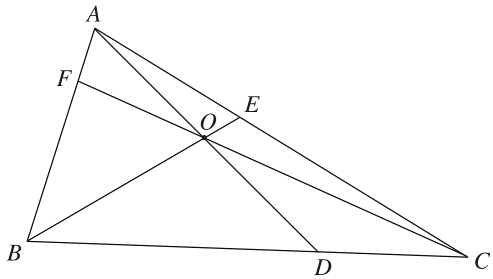

Figure 3 The case $(1, 2, 5)$

Extending the constructions

The previous constructions suggest the following general situation (using the notation from FIGURE 1). Let ABC be a triangle and let D be a point in BC such that $BD = n\,DC$, with $n > 0$. Now, let O be a point in AD such that $AO = m\,OD$, with $m > 0$. Finally, let BO and CO be cevians that determine points E and F, respectively.

If we assume that the area of ABC is 1, we can see that the area of AOC is $\frac{m}{(m+1)(n+1)}$, while the area of AOB is $\frac{mn}{(m+1)(n+1)}$. From this, it follows that

$$y = \frac{BO}{OE} = \frac{mn+n+1}{m},$$
$$z = \frac{CO}{OF} = \frac{m+n+1}{mn}.$$

If we take m, n to be be positive and rational, these considerations lead to the following set of positive rational solutions of equation (1), up to permutation:

$$(x, y, z) = \left(m, \frac{mn+n+1}{m}, \frac{m+n+1}{mn}\right), \quad m, n \in \mathbb{Q}^+.$$

In fact we can see that every positive rational solution of (1) is of this form.

PROPOSITION. *Every positive rational solution of equation* (1) *is of the form*

$$(x, y, z) = \left(m, \frac{mn + n + 1}{m}, \frac{m + n + 1}{mn}\right)$$

for some $m, n \in \mathbb{Q}^+$.

Proof. Let (x, y, z) be a positive rational solution of (1). Then, $xyz = x + y + z + 2$. This is equivalent (x being positive) to $(yx - 1)(zx - 1) = (x + 1)^2$. Then there are positive rational numbers m and n such that

$$m = x \text{ and } n = \frac{yx - 1}{x + 1} = \frac{x + 1}{zx - 1}.$$

It follows from the previous equalities that

$$y = \frac{mn + n + 1}{m}, \; z = \frac{m + n + 1}{mn}.$$

This provides us with a method to construct, given a positive rational solution (x, y, z) of equation (1), a triangle and a set of cevians leading to this solution. The steps are the following:

1. Draw any triangle ABC.
2. Compute $n = \frac{yx-1}{x+1} = \frac{x+1}{zx-1}$.
3. Find a point D in BC such that $BD = n\,DC$.
4. Put $m = x$ and find a point O in AD such that $AO = m\,OD$.

Another consequence of the last proposition is worth mentioning. For any two positive rational numbers x and y whose product is greater than 1, there is a positive rational number z such that (x, y, z) is a solution to equation (1), and hence there is a triangle whose concurrent cevians illustrate that solution. Given x and y, just take $z = \frac{x+y+2}{xy-1}$.

Finally, we leave to the reader the opportunity to pursue an extension of the topic. We have seen that non-positive integer solutions to equation (1) do not admit interesting geometric realizations. What about the non-positive rational solutions?

REFERENCES

1. E. J. Atzema and H. White, Euler on cevians, in *From Calculus to Computers: Using the Last 200 Years of Mathematics History in the Classroom*, Mathematical Association of America, Washington, DC, 2005, 75–92.
2. J. Benítez, A unified proof of Ceva and Menelaus' theorems using projective geometry, *J. Geom. Graph.* **11**(1) (2007) 39–44. http://www.heldermann.de/JGG/JGG11/JGG111/jgg11004.htm
3. N. A. Court, On the cevians of a triangle, *Nat. Math. Mag.* **18** (1943) 3–6. http://dx.doi.org/10.2307/3029911
4. J. P. Hogendijk, "Al-Mu'taman ibn Hūd, 11th century king of Saragossa and brilliant mathematician," *Historia Math.* **22**(1) (1995) 1–18. http://dx.doi.org/10.1006/hmat.1995.1001
5. J. Richter-Gebert, Meditations on Ceva's theorem, in *The Coxeter Legacy*, American Mathematical Society, Providence, RI, 2006, 227–254.
6. C. Thas, On some remarkable concurrences, *Forum Geom.* **2** (2002) 147–149. http://forumgeom.fau.edu/FG2002volume2/FG200218.pdf
7. V. Thébault, On cevians of a triangle, *Amer. Math. Monthly* **60** (1953) 167–173. http://dx.doi.org/10.2307/2307571

Summary Euler, in a paper published in 1812, studied in detail the situation when three cevians of a triangle were concurrent. In particular, he found a relation between the ratios in which the common point divided each of the cevians. This relation can be seen as an equation in three unknowns (the three ratios). In this paper we find the integer and rational solutions of this equation, providing explicit constructions and geometric interpretations in the positive cases.

The Volume of the Unit n-Ball

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Using calculus, we can compute that the n-dimensional volume, V_n, of the ball of radius 1 centered at the origin in n-dimensional Euclidean space—the unit n-ball—is given by

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} = \begin{cases} \dfrac{\pi^m}{m!} & \text{if } n = 2m, \\[2ex] \dfrac{2^{2m+1}\,\pi^m\, m!}{(2m+1)!} & \text{if } n = 2m+1. \end{cases}$$

In volume 62 of this MAGAZINE, Smith and Vamanamurthy [**3**] derive this formula in three different ways.

Given this formula, it follows from Stirling's approximation, $n! \sim \sqrt{2\pi n}\, e^{-n}\, n^n$, that

$$\text{the volume of the unit } n\text{-ball approaches 0 as } n \text{ approaches } \infty. \tag{1}$$

This surprising fact suggests the question: Is there a *simple* argument that allows us to see that V_n approaches 0? In this article, we give two simple arguments.

Cutting corners

One simple proof of (1) relies on covering the unit n-ball by simplices. First, using the Cauchy–Schwarz inequality (or in many other ways), we can prove that for positive numbers $x_1, \ldots, x_n$, if $x_1^2 + \cdots + x_n^2 \le 1$, then $x_1 + \cdots + x_n \le \sqrt{n}$. It follows that the unit n-ball is a subset of the polyhedral set

$$P_n = \left\{(x_1, x_2, \ldots, x_n) : \textstyle\sum_{i=1}^n |x_i| \le \sqrt{n}\,\right\}.$$

In turn, the polyhedron P_n is the union of 2^n congruent copies of the n-simplex

$$\Delta_n = \left\{x = (x_1, x_2, \ldots, x_n) : 0 \le x_i,\ \textstyle\sum_{i=1}^n x_i \le \sqrt{n}\,\right\}.$$

Later in this article, we show how to compute $|\Delta_n|$, the n-dimensional volume of Δ_n, with or without calculus. The result is

$$|\Delta_n| = \frac{n^{n/2}}{n!}. \tag{2}$$

Thus we obtain the estimate

$$V_n \le |P_n| = 2^n\, |\Delta_n| = \frac{2^n\, n^{n/2}}{n!}. \tag{3}$$

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Stirling's approximation tells us that the $n!$ in the denominator on the right-hand side of (3) overwhelms the numerator, so (1) follows.

Since Δ_n is the corner cut from the cube $\left[0,\ \sqrt{n}\right]^n$ by the plane $\sum_{i=1}^n x_i = \sqrt{n}$, we can think of our approach in this section as "cutting corners."

Thin boxes

It is possible to avoid the use of both calculus and Stirling's approximation in the "cutting corners" proof of (1), but to do so is unnatural. In this section, we give another simple, direct proof of (1)—a proof for which neither calculus nor Stirling's approximation can serve any purpose. The key observation is the following:

PROPOSITION. *For $n \geq 4$, the unit n-ball is contained in the union of $\binom{n}{4}$ congruent copies of the rectangular solid* (*see* FIGURE 1)

$$[-1,\ 1]^4 \times \left[-\frac{1}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right]^{n-4}. \tag{4}$$

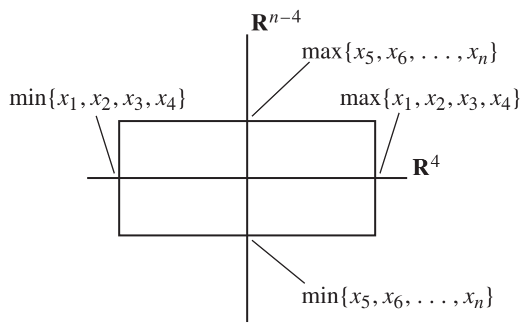


Figure 1 The rectangular solid in (4)

Proof. Since the result is obvious for $n = 4$, assume that $n \geq 5$ and let $(x_1, \ldots, x_n)$ be an arbitrary point in the unit n-ball. Partition the indices $1, 2, \ldots, n$ into two subsets (big coordinates versus small coordinates):

$$B = \left\{i : |x_i| > 1/\sqrt{5}\ \right\}$$

and

$$S = \left\{i : |x_i| \leq 1/\sqrt{5}\ \right\}.$$

Observe that if B were to contain five or more elements, then

$$|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2} > \sqrt{5\left(\frac{1}{\sqrt{5}}\right)^2} = 1$$

would hold, which would be a contradiction.

Thus we see that the unit n-ball is contained in the union of the rectangular solids formed as follows: For each partition λ of $\{1, 2, \ldots, n\}$ into $\{j_1, j_2, j_3, j_4\}$ and $\{k_1, k_2, \ldots, k_{n-4}\}$, set R_λ equal to the cartesian product of closed intervals

$$R_\lambda = I_1 \times I_2 \times \cdots \times I_n,$$

where

$$I_{j_\ell} = [-1,\ 1], \quad \text{for } \ell = 1, 2, 3, 4,$$
$$I_{k_m} = \left[-\frac{1}{\sqrt{5}},\ \frac{1}{\sqrt{5}}\right], \quad \text{for } m = 1, 2, \dots, n-4.$$

Now each R_λ is congruent to the rectangular solid in (4), and there are $\binom{n}{4}$ such R_λ's. ■

COROLLARY. *For $n \geq 4$, the n-dimensional volume of the unit n-ball is bounded above by*

$$\frac{25}{24}\,\frac{n^4}{\left(\sqrt{5}/2\right)^n}. \tag{5}$$

Moreover, the conclusion (1) *follows from the upper bound* (5).

Proof. The rectangular solid in (4) has n-dimensional volume

$$2^4\left(2/\sqrt{5}\right)^{n-4} = \frac{25}{\left(\sqrt{5}/2\right)^n},$$

so the unit n-ball is contained in the union of $\binom{n}{4}$ rectangular solids having that same n-dimensional volume. Bounding $\binom{n}{4}$ from above by $\frac{n^4}{24}$, we see that (5) is an upper bound for the n-dimensional volume of the unit n-ball.

Because the exponential denominator in (5) grows faster than the polynomial numerator, we see that (1) is true. ■

The rectangular solids R_λ can be called "thin boxes" because the n-dimensional volume of each is

$$\frac{25}{\left(\sqrt{5}/2\right)^n},$$

a quantity that approaches 0 exponentially fast as n approaches ∞.

Volume of the n-simplex

An n-simplex can be thought of as a hyperpyramid over an $(n-1)$-simplex. A *hyperpyramid* is the figure generated by joining a point called the *apex* to the points of a figure in a hyperplane, called the *base*.

To find the n-dimensional volume, v, of a hyperpyramid, we use an orthogonal coordinate system with origin at the apex that is oriented so that the base lies in the hyperplane $x_n = h > 0$. The value h is thus the height of the apex above the hyperplane containing the base. Let b denote the $(n-1)$-dimensional volume of the base. Any hyperplane $x_n = t$, $0 < t < h$, cuts the hyperpyramid in a figure similar to the base: The linear dimensions of the similar figure are scaled by t/h from the linear dimensions of the base, so the $(n-1)$-dimensional volume of the similar figure is scaled by $(t/h)^{n-1}$ from the volume of the base. Consequently, we see that

$$v = b\int_0^h (t/h)^{n-1}\,dt = \frac{1}{n}\,hb.$$

(Our argument is similar to that given by Sir Maurice Kendall in his monograph on n-dimensional geometry [**1**].)

We can now find the formula (2) for the n-dimensional volume of Δ_n. Let T_n denote the n-simplex

$$T_n = \left\{x = (x_1, x_2, \ldots, x_n) : 0 \le x_i,\ \sum_{i=1}^n x_i \le 1\right\}$$

in which each of the edges along the coordinate axes has length 1. Now T_n is a hyperpyramid of height 1 constructed over the base T_{n-1}, so its volume is given by $\mathrm{Vol}_n(T_n) = \mathrm{Vol}_{n-1}(T_{n-1})/n$, or, using induction, $\mathrm{Vol}_n(T_n) = 1/n!$. Since Δ_n is obtained from T_n by scaling by a linear factor of $\sqrt{n}$, we have

$$|\Delta_n| = n^{n/2}\,|T_n|.$$

Volume of the n-simplex without calculus

Some n-simplices have the property that $n!$ congruent copies fit together nicely to form an n-cube, and thus the volume of such a simplex is equal to the volume of the n-cube divided by $n!$. For example, as σ ranges over all permutations of the integers $1, 2, \ldots, n$, the tetrahedra

$$\left\{(x_1, x_2, \ldots, x_n) : 0 \le x_{\sigma(1)} \le x_{\sigma(2)} \le \cdots \le x_{\sigma(n)} \le 1\right\}$$

are all congruent to each other, their union is the unit cube, and any two intersect only in faces.

On the other hand, there are also some simplices that cannot be combined with congruent copies to form a rectangular solid and that cannot be cut into finitely many pieces, which can be combined to form a rectangular solid. Nonetheless, if we take it as axiomatic that the volume of similar n-dimensional figures is proportional to the nth power of the similarity factor, then we can determine the volume without using calculus. The discussion in this section is part of the subject known as "scissor congruence"; a recent reference is Igor Pak's monograph [**2**], in particular, sections 15, 16, and 17.

FIGURE 2 illustrates how the tetrahedron

$$\Delta = \left\{(x_1, x_2, x_3) : 0 \le x_i,\ \sum_{i=1}^3 x_i \le 1\right\}$$

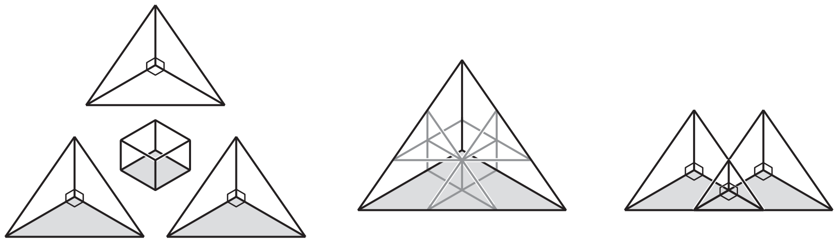
shown in FIGURE 2(b) can be assembled using the cube of side length $\frac{1}{3}$ and three tetrahedra congruent to $\frac{2}{3}\Delta$ shown in FIGURE 2(a). To find the volume, solve the equation

$$|\Delta| = \tfrac{1}{27} + 3|\tfrac{2}{3}\Delta| - 3|\tfrac{1}{3}\Delta| = \tfrac{1}{27} + 3\left(\tfrac{2}{3}\right)^3|\Delta| - 3\left(\tfrac{1}{3}\right)^3|\Delta| \qquad (6)$$

for $|\Delta|$. Equation (6) takes account of the fact that, when assembled as in FIGURE 2(b), each pair of tetrahedra congruent to $\frac{2}{3}\Delta$ intersects in a tetrahedron congruent to $\frac{1}{3}\Delta$; for example, FIGURE 2(c) shows how the bottom pair of tetrahedra congruent to $\frac{2}{3}\Delta$ intersect in FIGURE 2(b). The process illustrated for Δ in FIGURE 2 generalizes to n dimensions, but it becomes combinatorially more complex.

Concluding remarks

The referees have brought to my attention interesting material available on the internet. Arguments similar to "cutting corners" and "thin boxes" can be found at the link [**4**]. There are other, more complicated, arguments there as well. As with many internet sources, authorship is difficult to determine, so I am unable to acknowledge individuals. I am not aware that the arguments are in print, in the traditional sense.



(a) (b) (c)

Figure 2 Decomposing a tetrahedron into a cube and similar tetrahedra

REFERENCES

1. Maurice G. Kendall, *A Course in the Geometry of n-Dimensions,* Hafner Publishing, New York, 1961.
2. Igor Pak, *Lectures on Discrete and Polyhedral Geometry,* available at http://www.math.ucla.edu/~pak/book.htm (accessed Mar. 14, 2013).
3. David J. Smith and Mavina K. Vamanamurthy, How small is a unit ball? *Math. Mag.* **62** (1989) 101–107. http://dx.doi.org/10.2307/2690391
4. http://mathoverflow.net/questions/8258/whats-a-nice-argument-that-shows-the-volume-of-the-unit-n-ball-in-rn-approaches (accessed Dec. 3, 2012).

Summary Two simple proofs are given for the fact that the volume of the unit ball in n-dimensional Euclidean space approaches 0 as n approaches ∞. (Some authors use the term "unit sphere" for what is here called the unit ball.) One argument involves covering the unit ball by simplices. The other argument involves covering the unit ball by rectangular solids.

Ramanujan, Quadratic Forms, and the Sum of Three Cubes

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We all know that $3^2 + 4^2 = 5^2$. But did you know that $3^3 + 4^3 + 5^3 = 6^3$? Certainly many of you know this identity, but I must confess that I was unaware of it until one of my students showed it to me a few years ago. I was 53 at the time! In this note, we use this identity as a seed for finding other solutions of the Diophantine equation

$$A^3 + B^3 + C^3 = D^3. \tag{1}$$

The solutions we find are among the nicest of all solutions, the ones in which the integers A, B, C, D are each given by quadratic forms, that is, polynomials of the form $au^2 + buv + cv^2$.

Formulas for all solutions of Pythagorean Triples $a^2 + b^2 = c^2$ have been known since the time of Euclid, and partial solutions such as Plato's identity, $(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2$, were known much earlier. In 1915, Carmichael [**2**, pp. 38–43] found a four-parameter solution for all Pythagorean Quadruples, $a^2 + b^2 + c^2 = d^2$. According to Robert Spira [**10**], his proof was completed by Dickson [**6**] in a presentation given at the 1920 International Congress of Mathematicians in Strasbourg. Spira, in a 1962 MONTHLY article, discovered a connection between $a^2 + b^2 + c^2 = d^2$ and the then-unsolved equation $A^4 + B^4 + C^4 = D^4$, and the latter equation has since been solved by Noam Elkies [**7**] in 1988. In [**1**], Barbeau gives several formulas for the Diophantine equation $A^3 + B^3 + C^3 = D^3$, some of them given by quadratic forms, without saying how they are derived. Recently, Titus Piezas III [**9**] showed that Ramanujan's identity, given below, is a "particular case of a more general identity."

In 1754, Euler [**8**] found a formula for all *rational* solutions to the Diophantine equation (1). Euler's formula uses five parameters to describe the four numbers A, B, C, and D. The formula sometimes needs a non-integer rational as a parameter value to give a particular integer solution. Much more recently, Ajai Choudhry [**3**] found a five-parameter solution, using only integers, for all integer solutions to (1). More on the history of cubic identities can be found, of course, in Dickson [**5**, 550–561].

If we allow positive and negative solutions, then our Diophantine equation can be written as $A^3 + B^3 = C^3 + D^3$, or, in general, as $A^3 \pm B^3 \pm C^3 \pm D^3 = 0$. Since our equation is homogeneous, all rational solutions are just scaled-down integer solutions and vice-versa. The purpose of this note is much more modest than what Euler and Choudhry accomplished. Using only basic precalculus tools, we will show how to generate quadratic-form solutions to equation (1).

In the *Journal of the Indian Mathematical Society,* question 441 [**4**, pp. 15–26], Ramanujan gave the following quadratic form formula for our equation:

$$(3u^2 + 5uv - 5v^2)^3 + (4u^2 - 4uv + 6v^2)^3 + (5u^2 - 5uv - 3v^2)^3$$
$$= (6u^2 - 4uv + 4v^2)^3,$$

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and he asked us to *"find other quadratic relations satisfying similar relations."* Setting $u = 1$ and $v = 0$ in Ramanujan's formula gives us our original numerical identity $3^3 + 4^3 + 5^3 = 6^3$.

Finding quadratic-form solutions

We begin with the identity, $3^3 + 4^3 + 5^3 = 6^3$, as a seed to generate quadratic-form formulas to equation (1). In general, a single seed will generate at least four formulas. The only tools we need are cubing binomials, the quadratic formula, and, surprisingly, Diophantus' method of making a quadratic a square [**5**, p. 346].

Since cubics are considerably more difficult to work with than quadratics, we would like to set up an equation in two variables that, when simplified, eliminates the cubic terms. One way to do this is to look for rational solutions to the following equation

$$(3 + x)^3 + (4 + y)^3 + (5 - x)^3 = (6 + y)^3. \tag{2}$$

This equation has the advantage that when we expand the binomials, the cubic terms vanish. The cubes of integers vanish because of the seed identity, and the cubes of x and y vanish because of our choice of signs in (2). Therefore, when we expand the binomials, we obtain

$$6y^2 + 60y - 24x^2 + 48x = 0,$$

or

$$y^2 + 10y - 4x^2 + 8x = 0.$$

We now use the quadratic formula to solve for y:

$$y = \pm\sqrt{4x^2 - 8x + 25} - 5.$$

Since we are interested in rational solutions, we need to solve the equation

$$4x^2 - 8x + 25 = z^2$$

for rational numbers z. To do this, we use Diophantus' method of making a quadratic a square. (More on this method in a moment.) The method begins by noting that there is a rational number r such that $z = 2x + r$.

Thus, $4x^2 - 8x + 25 = (2x + r)^2 = 4x^2 + 4rx + r^2$. Our next step is to solve for x, giving $x = (25 - r^2)/(4r + 8)$. We now take the positive square root:

$$\sqrt{4x^2 - 8x + 25} = \frac{2r^2 + 8r + 50}{4r + 8}.$$

Hence, one solution for y is:

$$y = \sqrt{4x^2 - 8x + 25} - 5 = \frac{2r^2 - 12r + 10}{4r + 8}.$$

When we substitute $x = \frac{25-r^2}{4r+8}$ and $y = \frac{2r^2-12r+10}{4r+8}$ into

$$(3 + x)^3 + (4 + y)^3 + (5 - x)^3 = (6 + y)^3$$

we obtain

$$\left(\frac{-r^2+12r+49}{4r+8}\right)^3+\left(\frac{2r^2+4r+42}{4r+8}\right)^3+\left(\frac{r^2+20r+15}{4r+8}\right)^3$$
$$=\left(\frac{2r^2+12r+58}{4r+8}\right)^3.$$

Recall that r is rational, say $r = u/v$. Since $(4r + 8)$ is a common denominator for each of our expressions, substituting $r = u/v$ gives a new common denominator, $(4uv + 8v^2)^3$. When we clear the common denominator from the equation, we obtain the following quadratic-form representation for our Diophantine Equation:

$$\begin{aligned} A &= 49v^2 + 12uv - u^2,\\ B &= 42v^2 + 4uv + 2u^2,\\ C &= 15v^2 + 20uv + u^2,\\ D &= 58v^2 + 12uv + 2u^2. \end{aligned} \tag{3}$$

This is our first quadratic-form solution to (1). There are many more.

We could have taken at least two different paths, starting with the quadratic equation $y^2 + 10y - 4x^2 + 8x = 0$. We could have, just as well, solved for x in this quadratic:

$$x = \frac{1}{2}\left(2 \pm \sqrt{y^2+10y+4}\right).$$

The steps described here yield the following formula for A, B, C, and D:

$$\begin{aligned} A &= -u^2 - 6uv + 76v^2,\\ B &= 2u^2 - 16uv + 72v^2,\\ C &= u^2 - 26uv + 84v^2,\\ D &= 2u^2 - 24uv + 112v^2. \end{aligned} \tag{4}$$

Diophantus's method for making a quadratic a square

Diophantus' method is as follows. Given a quadratic of the form $a^2x^2 + bx + c$ or $ax^2 + bx + c^2$, we can find all rational numbers x such that the quadratic is the square of a rational number.

In the first case, we solve for x in the equation $a^2x^2 + bx + c = (ax + r)^2$. In the second case, we solve $ax^2 + bx + c^2 = (rx + c)^2$. Either way, x turns out to be a rational expression in the rational number r. All rational numbers x that yield squares of rationals arise in this way.

Variations

Did we get lucky with this particular seed? Sort of. Given any particular solution to our equation, say, $a^3 + b^3 + c^3 = d^3$, solving for either x or y in the equation $(a + x)^3 + (b + y)^3 + (c - x)^3 = (d + y)^3$, the quadratic under the radical always has a square for its constant term. In our example, we were lucky in that the coefficient of the squaring term was also a square. We used $z = 2x + r$ rather than $z = rx + 5$, in part because the resulting quadratic forms were a little simpler. But Diophantus's method would have served us in either case.

Do we need a seed for this method to work? No, but without a seed you need some luck and/or perseverance. For example, the equation

$$(3-x)^3 + (1+x)^3 + (1+y)^3 = (2+y)^3$$

generates the following identity:

$$\begin{aligned}(9u^2 - 11uv + v^2)^3 &+ (15u^2 - 5uv - v^2)^3 + (12u^2 - 4u + 2v^2)^3 \\ &= (18u^2 - 8uv + 2v^2)^3.\end{aligned}$$

We also could permute the x's and y's in our initial equation to generate other quadratic forms, e.g., $(3+x)^3 + (4-x)^3 + (5+y)^3 = (6+y)^3$.

Unlike in Ramanujan's formula, the seed $3^3 + 4^3 + 5^3 = 6^3$ is not readily evident in the first formula (3) that we derived. This can be remedied by using a linear transformation of the variables u and v. Going back to the equation $(3+x)^3 + (4+y)^3 + (5-x)^3 = (6+y)^3$, we see that our seed occurs if and only if $x = y = 0$, and this occurs if and only if $u/v = r = 5$. Now $u/v = 5$ implies that $u - 5v = 0$ or $-u + 5v = 0$. This suggests the linear transformation $u \mapsto -u + 5v$, $v \mapsto v$. If we use this transformation on formula (3), we see that the coefficients on the v^2 terms would be $84 = 3 \times 28$, $112 = 4 \times 28$, $140 = 5 \times 28$, $168 = 6 \times 28$. The reason we have a common factor of 28 is that when $r = 5$ in equation (3), the common denominator, $4r + 8$, is 28. Hence, when the fractions were cleared in that equation, we effectively multiplied each term by 28. One would be tempted to divide each of these transformed equations by 28, but this is not practical, since the coefficients of the u^2 and the uv terms are *not* multiples of 28. It turns out that any linear transformation of the form $u \mapsto au + 5v$, $v \mapsto v$ gives us coefficients of (84, 112, 140, 168) for the v^2 terms. (This tedious task can be made easier by using an AOS such as the DERIVE system that is installed on TI-89s or TI-92s.)

Upon examining the four new transformed formulas, we see that the parameter a must be a multiple of 14 if we want all the coefficients to be multiples of 28. When we use the transformation $u \mapsto 14u + 5v$, $v \mapsto v$, and then divide each of the formulas by 28, we obtain the following quadratic form formula for $A^3 + B^3 + C^3 = D^3$:

$$\begin{aligned}A &= -7u^2 + uv + 3v^2, \\ B &= 14u^2 + 12uv + 4v^2, \\ C &= 7u^2 + 15uv + 5v^2, \\ D &= 14u^2 + 16uv + 6v^2. \qquad (5)\end{aligned}$$

After all the work that has been done in deriving these formulas, we can enjoy the fruits of our labor by running through a couple of examples.

EXAMPLE 1. Setting $u = 1$ and $v = 2$ in (5) gives us $7^3 + 54^3 + 57^3 = 70^3$.

EXAMPLE 2. Setting $u = -3$ and $v = 1$ in (4) gives us

$$4^3 + 48^3 + (-36)^3 = 40^3$$

or

$$4^3 + 48^3 = 36^3 + 40^3.$$

Now divide both sides by 4^3:

$$1^3 + 12^3 = 9^3 + 10^3 = 1729.$$

This is G. H. Hardy's taxicab number, which, as Ramanujan knew, is the smallest number that is the sum of two cubes in two different ways.

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REFERENCES

1. Edward J. Barbeau, *Power Play,* Mathematical Association of America, Washington, DC, 1997.
2. R. D. Carmichael, *Diophantine Analysis,* Wiley, New York, 1915.
3. A. Choudhry, On equal sums of cubes, *Rocky Mountain Journal of Mathematics* **28**(4) (Winter, 1998) 1251–1257. http://dx.doi.org/10.1216/rmjm/1181071714
4. Bruce C. Berndt, Youn-Seo Choi, and Soon-Yi Kang, The problems submitted by Ramanujan to the Journal of the Indian Mathematical Society. In *Continued Fractions: From Analytic Number Theory to Constructive Approximation*, Bruce C. Berndt and Fritz Gesztesy, eds., American Mathematical Society, Providence, RI, pp. 15–56. http://dx.doi.org/10.1090/conm/236/03488
5. L. E. Dickson, *History of the theory of numbers,* vol. II, American Mathematical Society, Providence, RI, 2002.
6. L. E. Dickson, *Some Relations between the Theory of Numbers and Other Branches of Mathematics,* E. Privat, Toulouse, 1920.
7. N. Elkies, On $A^4 + B^4 + C^4 = D^4$, *Mathematics of Computation* **51** (1988) 825–835.
8. Leonhard Euler, Solutio generalis quorundam problematum Diophanteorum, quae vulgo nonnisi solutiones speciales admittere videntur, section 16, Euler archive, E255.
9. Titus Piezas, Ramanujan and the cubic equation $3^3 + 4^3 + 5^3 = 6^3$. http://www.oocities.org/titus_piezas/RamCube.pdf
10. R. Spira, The Diophantine equation $x^2 + y^2 + z^2 = m^2$, *American Mathematical Monthly* **69** (1962) 360–365. http://dx.doi.org/10.2307/2312125

Summary This article shows how, using only basic precalculus tools and a seed such as $3^3 + 4^3 + 5^3 = 6^3$, to generate quadratic-form formulas for the Diophantine equation $A^3 + B^3 + C^3 = D^3$.

Finite Sums of the Alcuin Numbers

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Alcuin of York, who died in 804 CE, was an English scholar and a leading figure in Charlemagne's court. He enjoyed posing problems involving counting, and compiled fifty-three such problems in a book, *Propositiones ad acuendos iuvenes*. One of those problems resulted in a sequence of numbers named after him. A recent paper in the *American Mathematical Monthly* by Bindner and Erickson [**6**] describes this history and introduces the Alcuin numbers.

We write $t(n)$ for the nth Alcuin number, for $n \geq 0$. Bindner and Erickson show that $t(n)$ is equal to the number of incongruent integer triangles of perimeter n, where an integer triangle is one for which all sides have integer length. Here are the first few Alcuin numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t(n)$	0	0	0	1	0	1	1	2	1	3	2	4	3	5	4	7

Bindner and Erickson derive properties of the Alcuin sequence using combinatorial arguments, counting techniques, and recurrences, and they relate the Alcuin numbers to integer partitions. The interest in Alcuin numbers is part of the long history [**10**, p. 181] of the fascination that authors have held with geometric objects having integer-valued features. A small sample of more recent efforts include [**4, 5, 7, 8, 9, 13, 15, 16, 17**].

Recurrences have been derived for Alcuin numbers [**6, 16**], and the following explicit formulas for $t(n)$ have been obtained [**2, 3, 6**] using integer partitions:

$$t(n) = \left\| \frac{n^2}{12} \right\| - \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n+2}{4} \right\rfloor = \begin{cases} \left\| n^2/48 \right\| & \text{if } n \text{ is even,} \\ \left\| (n+3)^2/48 \right\| & \text{if } n \text{ is odd,} \end{cases} \tag{1}$$

where $\|x\|$ stands for the integer closest to x, and $\lfloor x \rfloor$ is the floor, the largest integer that is less than or equal to x. An earlier detailed exposition of formula (1) appears in Honsberger's compilation [**14**, pp. 39–47] and is based on Andrews' works on partitions. In this paper we will give an alternate derivation using generating functions.

Formulas such as (1) do not lend themselves very easily to operations such as summation. For example, suppose that we want to find the number of incongruent integer triangles having perimeter *at most* n. Then we must compute the sum

$$s(n) = \sum_{k=0}^{n} t(k),$$

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which is formally awkward if we start with (1). How many of these triangles have even perimeter up to $2n$? That answer requires evaluation of

$$b(n) = \sum_{k=0}^{n} t(2k),$$

which presents the same difficulties.

There are standard collections of sums such as [**1, 11, 12**], but they contain formulas for mostly infinite sums; finite sums can be very difficult to determine. This is because many methods for finding closed-form formulas for infinite sums do not apply to finite sums. The generating function approach that we describe partially fills this gap.

Our purpose here is to demonstrate the generating function approach. By knowing the generating function for the Alcuin numbers $t(n)$, we can

- extract the closed-form formula (1) for $t(n)$;
- derive generating functions for finite sums, including $s(n)$ and $b(n)$; and
- use these to extract closed-form formulas for $s(n)$ and $b(n)$.

We also exploit the relationship between the Alcuin numbers and partitions of integers using generating functions. The approach illustrates a natural relation between different areas in mathematics such as number theory, calculus, and combinatorics, and in particular, between discrete mathematics and continuous analysis. Our paper concludes with a curious explicit formula for finite sums of floor functions.

Extracting the *n*th coefficient

We can associate a function with a sequence of (real or complex) numbers $\{a_n\}_{n\geq 0}$ in several ways. The most natural one is the formal power series:

$$A(z) = \sum_{n\geq 0} a_n z^n,$$

which is known as the *ordinary generating function* for the given sequence [**18**]. The nth coefficient a_n of this series is denoted by $[z^n]A(z)$. For example, if $A(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots$, then $[z^3]A(z) = 1/6$.

For the Alcuin numbers, the generating function $A(z)$ for $\{t(n)\}$ is known to be

$$A(z) = \sum_{n\geq 0} t(n)z^n = \frac{z^3}{(1-z^2)(1-z^3)(1-z^4)}. \tag{2}$$

Bindner and Erickon give a nice derivation of this by showing that $t(n)$ is equal to the number of partitions of $n-3$ as a sum of 2's, 3's, and 4's.

By expanding this generating function into a sum of partial fractions and then equating $t(n)$ with the sum of the nth coefficients of each part, we can obtain a closed-form formula for $t(n)$. The first step is to factor the denominator,

$$A(z) = \frac{z^3}{(1-z)^3(1+z)^2(1+z^2)(1+z+z^2)}.$$

We desire linear factors, so we use complex numbers to factor $(1+z^2) = (1+iz)(1-iz)$ and $1+z+z^2 = (z+\alpha_1)(z+\alpha_2)$, where

$$\alpha_1 = \frac{1+i\sqrt{3}}{2} = e^{i\pi/3}, \qquad \alpha_2 = \frac{1-i\sqrt{3}}{2} = e^{-i\pi/3}.$$

The partial fraction decomposition of $A(z)$ is found using a computer algebra system:

$$A(z) = \left(\frac{1}{24}\right)\frac{1}{(1-z)^3} - \left(\frac{13}{288}\right)\frac{1}{1-z} - \left(\frac{1}{16}\right)\frac{1}{(1+z)^2} - \left(\frac{1}{32}\right)\frac{1}{1+z} - \left(\frac{1}{8}\right)\frac{1+z}{1+z^2} + \left(\frac{1}{9}\right)\frac{2+z}{1+z+z^2}, \tag{3}$$

where

$$\frac{1+z}{1+z^2} = \left(\frac{1+i}{2}\right)\frac{1}{1+iz} + \left(\frac{1-i}{2}\right)\frac{1}{1-iz},$$
$$\frac{2+z}{1+z+z^2} = \frac{1}{(1+z\alpha_2)} + \frac{1}{(1+z\alpha_1)}.$$

Our method of extracting coefficients uses the geometric series formula $\frac{1}{1-bz} = \sum_{n\geq 0}(bz)^n$. Thus, $[z^n]\,\frac{1}{1-bz} = b^n$. Extracting coefficients in this manner, we obtain

$$[z^n]\,\frac{1+z}{1+z^2} = \frac{1+i}{2}\,(-i)^n + \frac{1-i}{2}\,i^n = (-1)^{\lfloor n/2\rfloor},$$
$$[z^n]\,\frac{2+z}{1+z+z^2} = (-\alpha_2)^n + (-\alpha_1)^n.$$

Here the expression $(-1)^{\lfloor n/2\rfloor}$ represents the repeating pattern of extracted coefficients $1, 1, -1, -1$. Notice that

$$\frac{1}{(1-z)^3} = \frac{1}{2}\frac{d^2}{dz^2}\frac{1}{1-z}, \qquad \frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z}, \qquad \frac{1}{(1+z)^2} = -\frac{d}{dz}\frac{1}{1+z}.$$

We use these equations to extract more coefficients:

$$[z^n]\,\frac{1}{(1-z)^3} = \frac{1}{2}\,[z^n]\,\frac{d^2}{dz^2}\frac{1}{1-z} = \frac{1}{2}\,[z^n]\,\frac{d^2}{dz^2}\sum_{k\geq 0} z^k = \frac{1}{2}\,[z^n]\sum_{k\geq 2} k(k-1)\,z^{k-2} = \frac{(n+2)(n+1)}{2},$$

$$[z^n]\,\frac{1}{(1-z)^2} = [z^n]\,\frac{d}{dz}\frac{1}{1-z} = [z^n]\,\frac{d}{dz}\sum_{k\geq 0} z^k = [z^n]\sum_{k\geq 1} k\,z^{k-1} = n+1,$$

$$[z^n]\,\frac{1}{(1+z)^2} = -[z^n]\,\frac{d}{dz}\frac{1}{1+z} = -[z^n]\,\frac{d}{dz}\sum_{k\geq 0}(-z)^k = [z^n]\sum_{k\geq 1} k(-1)^{k-1}\,z^{k-1} = (-1)^n(n+1).$$

Summing all coefficients from (3) and confirming that $\alpha_1^n + \alpha_2^n = 2\cos\frac{n\pi}{3}$, we arrive at the following closed-form formula:

$$t(n) = \frac{1}{48}(n+1)(n+2) - \frac{(-1)^n}{16}(n+1) - \frac{13}{288} - \frac{(-1)^n}{32} - \frac{(-1)^{\lfloor n/2 \rfloor}}{8} + (-1)^n \frac{2}{9}\cos\left(\frac{n\pi}{3}\right). \tag{4}$$

We can replace $(-1)^{\lfloor n/2 \rfloor}$ by $\cos(\frac{n\pi}{2}) + \sin(\frac{n\pi}{2})$. This exact formula for $t(n)$ leads directly to (1). In this regard, notice that the first two summands in (4) add to $(n^2-1)/48$ or $((n+3)^2-4)/48$, depending on whether n is even or odd respectively, and the remaining terms add to a number no greater than $1/3$ in magnitude (actually, the remaining sum is periodic with minimum value $-5/16$ and maximum value $1/3$).

The foregoing computation illustrates the basic extraction technique.

Generating functions and sums

Turning to finite sums, there is a general method to find a formula for the sum $s(n) = \sum_{k=0}^{n} a_k$ of any sequence $\{a_k\}$. We begin with the generating function $A(z) = \sum_{k\geq 0} a_k z^k$ for the sequence, multiply it by $\frac{1}{1-z}$, which produces the generating function for the sequence $\{s(n)\}$ of sums. That is,

$$s(n) = \sum_{k=0}^{n} a_k = [z^n]\,\frac{A(z)}{1-z}, \qquad \text{where} \quad A(z) = \sum_{k\geq 0} a_k z^k. \tag{5}$$

This is true because the generating function for the sequence of partial sums $s(n) = \sum_{k=0}^{n} a_k$ is

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k\right) z^n = \sum_{k\geq 0}\sum_{n\geq k} a_k z^n = \sum_{k\geq 0} a_k \sum_{n\geq k} z^n = \sum_{k\geq 0} a_k\,\frac{z^k}{1-z} = \frac{1}{1-z}\sum_{k\geq 0} a_k\,z^k = \frac{A(z)}{1-z}.$$

This method works well when $A(z)$ is a rational function.

Alcuin partial sums

Let us return to the problem of finding the number of incongruent integer triangles having perimeter at most n. Referring to (2), the generating function for the sequence $\{s(n) = \sum_{k=0}^{n} t(k)\}$ is

$$\frac{A(z)}{1-z} = \frac{z^3}{(1-z)^4(1+z)^2(1+z^2)(1+z+z^2)},$$

which has partial fraction decomposition (found with the help of a computer):

$$\frac{A(z)}{1-z} = \frac{1}{54}\,\frac{3+i\sqrt{3}}{z+\alpha_1} + \frac{1}{54}\,\frac{3-i\sqrt{3}}{z+\alpha_2} + \frac{1}{16(i-z)} - \frac{1}{16(i+z)} - \frac{1}{32(1+z)} - \frac{1}{32(1+z)^2} - \frac{13}{288(1-z)} - \frac{13}{288(1-z)^2} + \frac{1}{24(1-z)^4}.$$

Before extracting coefficients, we rewrite the first two terms on the right-hand side of this equation in a form that facilitates the task. Using the relation $\alpha_1^{-1} = \alpha_2$, we have

$$\frac{1}{54}\,\frac{3+i\sqrt{3}}{z+\alpha_1} + \frac{1}{54}\,\frac{3-i\sqrt{3}}{z+\alpha_2} = \frac{\alpha_2}{54}\,\frac{3+i\sqrt{3}}{1+\alpha_2 z} + \frac{\alpha_1}{54}\,\frac{3-i\sqrt{3}}{1+\alpha_1 z}.$$

Extracting the nth coefficient of $A(z)/(1-z)$, we obtain

$$\begin{aligned} s(n) &= [z^n]\,\frac{A(z)}{1-z} \\ &= \alpha_2 \frac{3+i\sqrt{3}}{54}[z^n] \sum_{k\ge 0}(-1)^k\,(\alpha_2 z)^k + \alpha_1 \frac{3-i\sqrt{3}}{54}[z^n] \sum_{k\ge 0}(-1)^k\,(\alpha_1 z)^k \\ &\quad + \frac{1}{16i}[z^n] \sum_{k\ge 0}(-1)^k (iz)^k - \frac{1}{16i}[z^n] \sum_{k\ge 0}(iz)^k - \frac{1}{32}[z^n] \sum_{k\ge 0}(-1)^k z^k \\ &\quad + \frac{1}{32}[z^n]\,\frac{d}{dz} \sum_{k\ge 0}(-1)^k z^k - \frac{13}{288}[z^n] \sum_{k\ge 0} z^k - \frac{13}{288}[z^n]\,\frac{d}{dz} \sum_{k\ge 0} z^k \\ &\quad + \frac{1}{24}\,\frac{1}{3!}\,[z^n]\,\frac{d^3}{dz^3} \sum_{k\ge 0} z^k \\ &= \frac{3+i\sqrt{3}}{54}\,(-1)^n\,\alpha_2^{n+1} + \frac{3-i\sqrt{3}}{54}\,(-1)^n\,\alpha_1^{n+1} + \frac{1}{16}\,(-1)^n\,i^{n-1} \\ &\quad - \frac{1}{16}\,i^{n-1} - \frac{1}{32}\,(-1)^n + \frac{1}{32}\,(-1)^{n+1}\,(n+1) - \frac{13}{288} - \frac{13}{288}\,(n+1) \\ &\quad + \frac{(n+3)(n+2)(n+1)}{144}. \end{aligned}$$

Since $\alpha_1^n + \alpha_2^n = 2\,\cos\frac{n\pi}{3}$ and $\alpha_1^n - \alpha_2^n = 2i\,\sin\frac{n\pi}{3}$, we rewrite the sum as

$$\begin{aligned} s(n) = {} & \frac{(-1)^n}{9}\left(\cos\frac{(n+1)\pi}{3} + \frac{1}{\sqrt{3}}\,\sin\frac{(n+1)\pi}{3}\right) \\ & - \frac{1}{8}\,\sin\frac{n\pi}{2} - (-1)^n\,\frac{n+2}{32} - \frac{13(n+2)}{288} + \frac{(n+3)(n+2)(n+1)}{144}. \end{aligned}$$

Note that the expression $\cos\frac{(n+1)\pi}{3} + \frac{1}{\sqrt{3}}\,\sin\frac{(n+1)\pi}{3}$ is equal to $\frac{2}{\sqrt{3}}\,\sin\frac{(n+2)\pi}{3}$, and combining terms,

$$\begin{aligned} &\frac{(n+3)(n+2)(n+1)}{144} - \frac{13(n+2)}{288} - (-1)^n\,\frac{n+2}{32} \\ &\qquad = \frac{2n^3 + 12n^2 - (-1)^n 9(n+2) + 9n - 14}{288}. \end{aligned}$$

Thus, we can abbreviate our exact sum to

$$s(n) = \left\| \frac{2n^3 + 12n^2 + 9n - (-1)^n 9n}{288} \right\| = \begin{cases} \|(n^3 + 6n^2)/144\|, & \text{if } n \text{ is even,} \\ \|(n^3 + 6n^2 + 9n)/144\|, & \text{if } n \text{ is odd.} \end{cases}$$

We tabulate some values of $s(n)$ beginning with the first few nonzero Alcuin numbers.

n	3	4	5	6	7	8	9	10	11	12	20	500	1001
$t(n)$	1	0	1	1	2	1	3	2	4	3	8	5208	21000
$s(n)$	1	1	2	3	5	6	9	11	15	18	72	878,472	7,007,111

Even sums

How many integer triangles have even perimeter up to (but no larger than) $2n$? Let $\{b(n)\}$ be the sequence of sums of even-numbered Alcuin's numbers. Thus, $b(n) = \sum_{k=0}^{n} t(2k)$. First, we need to find the generating function for the sequence $\{t(2k)\}$; we compute $\frac{1}{2}[A(z) + A(-z)] = \sum_{n\geq 0} t(2n)\, z^{2n}$. This series contains only even powers of z. The generating function that we seek is then

$$B(z) = \frac{1}{2}\left[A(\sqrt{z}) + A(-\sqrt{z})\right] = \sum_{n\geq 0} t(2n)\, z^n = \frac{z^3}{(1-z)(1-z^2)(1-z^3)}. \tag{6}$$

According to (5), the generating function for the sequence $\{b(n)\}$ of sums is $\frac{B(z)}{1-z}$. Using a computer algebra system, we obtain

$$\frac{B(z)}{1-z} = \frac{1}{9}\frac{1+z}{1+z+z^2} + \frac{1}{6}\frac{1}{(1-z)^4} - \frac{1}{4}\frac{1}{(1-z)^3} - \frac{1}{72(1-z)^2} + \frac{7}{144(1-z)} - \frac{1}{16(1+z)},$$

where

$$\frac{1+z}{1+z+z^2} = \frac{i\alpha_2^2}{\sqrt{3}(1+z\alpha_2)} - \frac{i\alpha_1^2}{\sqrt{3}(1+z\alpha_1)}.$$

Extracting coefficients from this portion yields

$$[z^n]\frac{1+z}{1+z+z^2} = \frac{i}{\sqrt{3}}\left(\alpha_2^{2n+2} - \alpha_1^{2n+2}\right) = \frac{-4}{\sqrt{3}}\sin\frac{(2n+2)\pi}{3}.$$

Extracting the other coefficients and adding all terms, we arrive at

$$b(n) = \frac{2(-1)^n}{9\sqrt{3}}\sin\frac{(n+2)\pi}{3} - \frac{(-1)^n}{16} + \frac{7}{144} - \frac{(n+1)(9n+19)}{72} + \frac{(n+1)(n+2)(n+3)}{36}.$$

The last two terms simplify to $(2n^3 - 7 - 6n + 3n^2)/72$, from which we are able to abbreviate our exact formula for $b(n) = \sum_{k=0}^{n} t(2k)$:

$$b(n) = \sum_{k=0}^{n} t(2k) = \left\| \frac{2n^3 + 3n^2 - 6n}{72} \right\|.$$

Here are some values of the sum.

n	2	3	4	5	6	7	8	9	10	20	500	1001
$b(n)$	0	1	2	4	7	11	16	23	31	237	3,482,597	27,902,861

Partitions

It is known that Alcuin numbers are related to partitions of integers [**3, 6**]. In this final section, we explore this relationship. Let $p_r(n)$ denote the number of partitions of the nonnegative integer n into r positive integer parts (that is, summands), and let $\pi_r(n)$ be the number of partitions of n in which no part is greater than r. Thus,

$$\pi_r(n) = \sum_{j=1}^{r} p_j(n).$$

The generating function for $\pi_r(n)$ is (see [**3, 11, 18**])

$$\Pi_r(z) = \prod_{k=1}^{r} \frac{1}{1-z^k} = \sum_{k\geq 0} \pi_r(j) z^j.$$

Since $p_r(n) = \pi_r(n) - \pi_{r-1}(n)$, we can find the generating function $P_r(z)$ of the sequence $\{p_r(n)\}_{n\geq 0}$, by subtracting two generating functions as follows:

$$P_r(z) = \Pi_r(z) - \Pi_{r-1}(z) = \frac{z^r}{\prod_{k=1}^{r}(1-z^k)}.$$

For $r = 2$ and $r = 3$, the generating functions for the number of partitions of an integer n into two and three parts become

$$P_2(z) = \frac{z^2}{(1-z)(1-z^2)} \qquad \text{and} \qquad P_3(z) = \frac{z^3}{(1-z)(1-z^2)(1-z^3)},$$

respectively. Using a partial fraction decomposition, we have

$$P_2(z) = \frac{1}{2(1-z)^2} - \frac{3}{4(1-z)} + \frac{1}{4(1+z)},$$

$$P_3(z) = \frac{1}{6(1-z)^3} - \frac{1}{4(1-z)^2} - \frac{1}{72(1-z)} - \frac{1}{8(1+z)} + \frac{2+z}{9(1+z+z^2)},$$

which allows us to extract nth coefficients explicitly:

$$p_2(n) = [z^n]P_2(z) = \frac{n+1}{2} - \frac{3}{4} + (-1)^n \frac{1}{4} = \frac{n}{2} - \frac{1}{4} + (-1)^n \frac{1}{4} = \left\lfloor \frac{n}{2} \right\rfloor,$$

and

$$p_3(n) = [z^n]P_3(z) = \frac{n^2-1}{12} - \frac{1}{72} - (-1)^n \frac{1}{8} + (-1)^n \frac{2}{9} \cos \frac{n\pi}{3} = \left\| \frac{n^2}{12} \right\|. \qquad (7)$$

We have seen the function $P_3(z)$ before. It is equal to the generating function $B(z)$ of even numbered Alcuin numbers given in (6). Therefore, the Alcuin number $t(2n)$ is equal to the number of partitions of n into three parts:

$$\boxed{t(2n) = p_3(n).}$$

Bindner and Erickson present a specific bijection supporting this equation.

In general, the Alcuin number is related to $p_2(n)$ and $p_3(n)$ by the equation

$$t(n) = p_3(n) - \sum_{k=1}^{\lfloor n/2 \rfloor} p_2(k). \qquad (8)$$

To find the exact expression for the sum $\sum_{k=1}^{n} p_2(k)$, we use the partial fraction decomposition

$$\frac{P_2(z)}{1-z} = \frac{1}{2(1-z)^3} - \frac{3}{4(1-z)^2} + \frac{1}{8(1-z)} + \frac{1}{8(1+z)},$$

and then extract coefficients:

$$[z^n]\,\frac{P_2(z)}{1-z} = \sum_{k=1}^{n} p_2(k) = \frac{n^2-1}{4} + \frac{1}{8} + (-1)^n\,\frac{1}{8}.$$

As a byproduct, we get a nice formula:

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor = \frac{n^2}{4} - \frac{1}{8} + \frac{(-1)^n}{8} \qquad \Longrightarrow \qquad \sum_{k=1}^{\lfloor n/2 \rfloor} \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{n}{4} \right\rfloor \left\lfloor \frac{n+2}{4} \right\rfloor.$$

Thus, (7) and (8) yield formula (1) from the introduction.

REFERENCES

1. M. Abramowitz, M. Stegun and A. Irene A., eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1972.
2. G. E. Andrews, A note on partitions and triangles with integer sides, *American Mathematical Monthly* **86** (1979) 477–478. http://dx.doi.org/10.2307/2320420
3. G. E. Andrews, MacMahon's partition analysis: II fundamental theorems, *Annals of Combinatorics* **4** (2000) 327–338. http://dx.doi.org/10.1007/PL00001284
4. R. A. Beauregard, Diametric quadrilaterals with two equal sides, *College Mathematics Journal* **40** (2009) 17–21. http://dx.doi.org/10.4169/193113409X469668
5. R. A. Beauregard and E. R. Suryanarayan, Pythagorean boxes, *Mathematics Magazine* **74** (2001) 222–227. http://dx.doi.org/10.2307/2690724
6. D. J. Bindner and M. Erickson, Alcuin's sequence, *American Mathematical Monthly* **119** (2012) 115–121. http://dx.doi.org/10.4169/amer.math.monthly.119.02.115
7. R. H. Buchholz and J. A. MacDougall, Heron quadrilaterals with sides in arithmetic or geometric progression, *Bulletin of the Australian Mathematical Society* **59** (1999) 263–269. http://dx.doi.org/10.1017/S0004972700032883
8. B. Burn, Triangles with a 60° angle and sides of integer length, *Mathematical Gazette* **87** (2003) 148–153.
9. M. N. Deshpande, Some new triples of integers and associated triangles, *Mathematical Gazette* **86** (2002) 464–466. http://dx.doi.org/10.2307/3621143
10. L. Dickson, *History of the Theory of Numbers*, vol. 2, G. E. Stechert, New York, 1934.
11. V. A. Dobrushkin, *Methods in Algorithmic Analysis*, CRC Press, Boca Raton, FL, 2010.
12. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Alan Jeffrey and Daniel Zwillinger (eds.), 7th ed., Academic Press, Orlando, FL, 2007.
13. M. D. Hirschhorn, Commensurable Triangles, *Mathematical Gazette* **95** (2011) 61–63.
14. R. Honsberger, *Mathematical Gems III*, Mathematical Association of America, Washington, DC, 1985.
15. T. Jenkyns and E. Muller, Triangular triples from ceilings to floors, *American Mathematical Monthly* **107** (2000) 634–639. http://dx.doi.org/10.2307/2589119
16. J. H. Jordan, R. Walch, and R. J. Wisner, Triangles with integer sides, *American Mathematical Monthly* **86** (1979) 686–689. http://dx.doi.org/10.2307/2321300
17. N. Krier and B. Manvel, Counting integer triangles, *Mathematics Magazine* **71** (1998) 291–295. http://dx.doi.org/10.2307/2690701
18. H. Wilf, *Generatingfunctionology*, 3d ed., A.K. Peters, Wellesley, MA, 2006.

Summary The Alcuin number $t(n)$ is equal to the number of incongruent integer triangles having perimeter n, where n is an integer. Using generating functions, we give a derivation of well-known formulas for the Alcuin sequence $\{t(n)\}$ that involves the closest integer function $||x||$, and floor function $\lfloor x \rfloor$. These formulas do not lend themselves very easily to operations such as summation. To find the number of incongruent integer triangles having perimeter at most n, we must evaluate the sum $\sum_{k=0}^{n} t(k)$, or to find those with even perimeter up to $2n$, we must evaluate $\sum_{k=0}^{n} t(2k)$. These computations are theoretically awkward. In this article, we develop formulas in both closed form and abbreviated form for these sums using generating functions. In the process, we exploit the relationship between the Alcuin numbers and partitions of integers.

PROBLEMS

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PROPOSALS

To be considered for publication, solutions should be received by March 1, 2014.

1926. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let n be a positive integer. For $1 \leq i \leq n$, let x_i and y_i be positive real numbers such that $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$. Prove that there exist some rearrangements $X_1, \ldots, X_n$ of $x_1, \ldots, x_n$ and $Y_1, \ldots, Y_n$ of $y_1, \ldots, y_n$ such that $\sum_{i=1}^{k} X_i / Y_i \geq k$ for all $1 \leq k \leq n$.

1927. *Proposed by Azkar Dzhumadil'daev, Institute of Mathematics, Almaty, Kazakhstan.*

Let $p > 3$ be a prime number. Prove that the numerator of the fraction

$$\sum_{n=3}^{p-1} \frac{n^3}{(n-1)(n-2)}$$

is divisible by p.

1928. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let P be an arbitrary point inside $\triangle ABC$. Let α, β, and γ be the distances from P to the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Prove that

$$(PA + PB + PC)\left(\frac{PA}{\beta\gamma} + \frac{PB}{\alpha\gamma} + \frac{PC}{\alpha\beta}\right) \geq 36.$$

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We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1929. *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $a > 0$ be a real number and let (x_n) be the sequence defined by the recurrence relation $x_1 = 1$ and for $n \geq 1$,

$$x_{n+1} = x_n + an \prod_{i=1}^{n} x_i^{-1/n}.$$

(a) Prove that $\lim_{n\to\infty} x_n = \infty$.

(b) Calculate $\lim_{n\to\infty} x_n / \ln n$.

1930. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Find the value of

$$\sum_{n=2}^{\infty} (-1)^n n(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)),$$

where ζ denotes the Riemann zeta function defined by $\zeta(z) = \sum_{n=1}^{\infty} 1/n^z$ for $z \in \mathbb{C}$ with $\Re(z) > 1$.

Quickies

Answers to the Quickies are on page 294.

Q1033. *Proposed by Mowaffaq Hajja, Mathematics Department, Yarmouk University, Irbid, Jordan.*

Let $ABCD$ be a simple planar quadrilateral. Prove that there is a unique point P in the plane of $ABCD$ such that the triangles PAB and PCD are directly similar.

Q1034. *Proposed by Rick Mabry, LSU Shreveport, Shreveport, LA.*

In his book, *Amusements in Mathematics* (1917), Henry Dudeney considers "A Printer's Error" (Problem #115) in which $2^5 9^2$ is incorrectly formatted without superscripting the exponents, resulting in 2592. The amusement is the anomaly that $2^5 9^2 = 2592$ is actually true—a correct printer's error. See Erich Friedman's web page at http://www2.stetson.edu/~efriedma/mathmagic/0601.html for examples of 4-digit correct printer's errors and some open questions, including examples in other bases.

Find all correct 2-digit printer's errors of the form $a^b = ab_N$ for some base N. That is, find all integers N, a, and b such that $N \geq 2$, $1 \leq a < N$, $0 \leq b < N$, and $a^b = Na + b$.

Solutions

Lagrange interpolating and Rolle's Theorem **October 2012**

1901. *Proposed by Ángel Plaza and César Rodríguez, Department of Mathematics, Universidad de las Palmas de Gran Canaria, Las Palmas, Spain.*

Let $f : [a, b] \to \mathbb{R}$ be an n times continuously differentiable function on (a, b), and let $a_1, a_2, \ldots, a_{n+1}$ be $n + 1$ distinct numbers in (a, b). Prove that there exists c in (a, b) such that

$$\sum_{j=1}^{n+1} f(a_j) \prod_{\substack{1\le i\le n+1\\ i\neq j}} (a_j - a_i)^{-1} = \frac{f^{(n)}(c)}{n!}.$$

Solution by Robert L. Doucette, McNeese State University, Lake Charles, LA.
Let p be the interpolating polynomial of degree $\le n$ that passes through the points $(a_j, f(a_j))$ for $1 \le j \le n + 1$. In Lagrange form, p is given by

$$p(x) = \sum_{j=1}^{n+1} f(a_j) \prod_{\substack{1\le i\le n+1\\ i\neq j}} \frac{x - a_i}{a_j - a_i}.$$

Define $\phi(x) = f(x) - p(x)$. Then $\phi(a_j) = 0$ for $1 \le j \le n + 1$. By Rolle's Theorem, ϕ' has n distinct zeros in (a, b). Applying Rolle's Theorem again, ϕ'' has $n - 1$ distinct zeros in (a, b). We may continue in this way until we establish that $\phi^{(n)}$ has one zero $c \in (a, b)$. The nth derivative of p equals $n!$ times the coefficient of x^n. It follows that

$$\phi^{(n)}(c) = f^{(n)}(c) - n! \sum_{j=1}^{n+1} f(a_j) \prod_{\substack{1\le i\le n+1\\ i\neq j}} (a_j - a_i)^{-1} = 0.$$

Also solved by George Apostolopoulos (Greece), Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Bruce S. Burdick, Hongwei Chen, Lixing Han, John C. Kieffer, Omran Kouba (Syria), Elias Lampakis (Greece), Northwestern University Math Problem Solving Group, Tomas Persson and Mikael P. Sundqvist (Sweden), Skidmore College Problem Group, Traian Viteam (Chile), and the proposers.

Positive polynomials generated by a sequence **October 2012**

1902. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let $a_0, a_1, \ldots, a_{2n}$ be positive real numbers. Prove that there are at least $(n!)^2$ different permutations σ of $\{0, 1, \ldots, 2n\}$ such that

$$a_{\sigma(0)} + a_{\sigma(1)}x + \cdots + a_{\sigma(2n)}x^{2n} > 0$$

for all $x \in \mathbb{R}$.

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI.
Since there is no loss of generality, we may assume that $a_0 \le a_1 \le \cdots \le a_{2n}$. Let f be the polynomial of degree $2n$ in which every coefficient is a_n. Then $a_n(x^{2n+1} - 1) = (x - 1)f(x)$. Therefore, the roots of f are the nonreal $(2n + 1)$th roots of unity. In particular, f has no real roots and so $f(x) > 0$ for all $x \in \mathbb{R}$. Let g be a polynomial of degree $2n$ in which the coefficients of x^k for $k \in \{1, 3, \ldots, 2n - 1\}$ are some permutation of $a_0, a_1, \ldots, a_{n-1}$, and the coefficients of x^k for $k \in \{0, 2, 4, \ldots, 2n\}$ are some permutation of $a_n, a_{n+1}, \ldots, a_{2n}$. By comparing the kth degree terms of f and g for every k, and for $x < 0$, it follows that $f(x) \le g(x)$ for all $x < 0$. Therefore $g(x) > 0$ for all $x \in \mathbb{R}$. Since there are $n!(n + 1)! \ge (n!)^2$ ways to select g subject to the given criteria, the proof is complete.

Also solved by George Apostolopoulos (Greece), Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Con Amore Problem Group (Denmark), Chip Curtis, Daniele Fakhoury (Italy) and

Giulia Giovannotti (Italy), Marty Getz and Dixon Jones, Eugene A. Herman, Omran Kouba (Syria), Elias Lampakis (Greece), Peter McPolin (Northern Ireland), Nicholas C. Singer, and the proposer.

Rearranging an inequality **October 2012**

1903. *Proposed by Mihály Bencze, Brasov, Romania.*

Let x, y, $a_1 = a_4$, $a_2 = a_5$, and a_3 be positive real numbers. Prove that

$$8(a_1^3 + a_2^3 + a_3^3) \geq \sum_{k=1}^{3} \left[(a_k + a_{k+1})^x (a_k + a_{k+2})^y \right]^{\frac{3}{x+y}} \geq 3(a_1 + a_2)(a_2 + a_3)(a_3 + a_1).$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

Let $\alpha = (a_1 + a_2)^3$, $\beta = (a_2 + a_3)^3$, $\gamma = (a_3 + a_1)^3$, and $p = x/(x + y)$. Applying the Arithmetic Mean–Geometric Mean Inequality gives

$$\begin{aligned}\sum_{k=1}^{3} \left[(a_k + a_{k+1})^x (a_k + a_{k+2})^y \right]^{\frac{3}{x+y}} &= \alpha^p \gamma^{1-p} + \beta^p \alpha^{1-p} + \gamma^p \beta^{1-p} \\ &\geq 3(\alpha^p \gamma^{1-p} \beta^p \alpha^{1-p} \gamma^p \beta^{1-p})^{1/3} = 3(\alpha\beta\gamma)^{1/3} \\ &= 3(a_1 + a_2)(a_2 + a_3)(a_3 + a_1),\end{aligned}$$

which is the second inequality.

For the first inequality, assume without loss of generality that $a_1 \geq a_2 \geq a_3$. It follows that $\alpha \geq \gamma \geq \beta$. Since $0 < p < 1$, we have that

$$\alpha^p \geq \gamma^p \geq \beta^p \quad \text{and} \quad \alpha^{1-p} \geq \gamma^{1-p} \geq \beta^{1-p}.$$

Using the rearrangement inequality, we can write

$$\alpha^p \gamma^{1-p} + \beta^p \alpha^{1-p} + \gamma^p \beta^{1-p} \leq \alpha^p \alpha^{1-p} + \beta^p \beta^{1-p} + \gamma^p \gamma^{1-p} = \alpha + \beta + \gamma.$$

That is,

$$\sum_{k=1}^{3} \left[(a_k + a_{k+1})^x (a_k + a_{k+2})^y \right]^{\frac{3}{x+y}} \leq (a_1 + a_2)^3 + (a_2 + a_3)^3 + (a_3 + a_1)^3.$$

Finally, since $4(x^3 + y^3) - (x + y)^3 = 3(x + y)(x - y)^2$, we see that

$$(a_k + a_{k+1})^3 \leq 4(a_k^3 + a_{k+1}^3)$$

for $k = 1, 2, 3$. Thus

$$\begin{aligned}(a_1 + a_2)^3 + (a_2 + a_3)^3 + (a_3 + a_1)^3 &\leq 4(a_1^3 + a_2^3 + a_2^3 + a_3^3 + a_3^3 + a_1^3) \\ &= 8(a_1^3 + a_2^3 + a_3^3),\end{aligned}$$

which yields the first inequality.

Also solved by Arkady Alt, George Apostolopoulos (Greece), Michel Bataille (France), Berry College Dead Poets Society, Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robert Calcaterra, Minh Can, Con Amore Problem Group (Denmark), Chip Curtis, Marian Dincă (Romania), Robert L. Doucette, Dmitry Fleischman, Lixing Han and Luyuan Yu, Elias Lampakis (Greece), Paolo Perfetti (Italy), Nicholas C. Singer, Michael Vowe (Switzerland), Connie Xu, and the proposer.

Bounding the antidiagonal of an orthogonal projector **October 2012**

1904. *Proposed by Oskar Maria Baksalary, Adam Mickiewicz University, Poznań, Poland, and Götz Trenkler, Dortmund University of Technology, Dortmund, Germany.*

Let A be an $n \times n$ complex matrix of rank r, $n \geq 2$, and $0 < r \leq n$. Let s be the sum of the elements of the antidiagonal of A; that is, if $A = (a_{ij})$, then $s = a_{1,n} + a_{2,n-1} + \cdots + a_{n,1}$. Prove that if A is idempotent and Hermitian (i.e., an orthogonal projector), then $|s| < \sqrt{nr}$.

Solution by Nicholas C. Singer, Annandale, VA.
If $r = n$, then A is invertible. Hence $A = I_n$, by applying A^{-1} to the equation $A^2 = A$. Then, $s = 0$ if n is even and $s = 1$ if n is odd, and $1 < n = \sqrt{nr}$. Henceforth assume $1 \leq r < n$.

By the spectral theorem, $A = \sum_{k=1}^{r} u_k u_k^*$, where $\{u_1, \ldots, u_r\}$ is an orthonormal set of $\mathbb{C}^n$ vectors (i.e., an orthonormal basis for Range(A).) Then

$$s = \sum_{m=1}^{n} a_{m,n+1-m} = \sum_{m=1}^{n} \left(\sum_{k=1}^{r} u_k u_k^* \right)_{m,n+1-m} = \sum_{m=1}^{n} \sum_{k=1}^{r} (u_k)_m \overline{(u_k)_{n+1-m}}$$
$$= \sum_{k=1}^{r} \sum_{m=1}^{n} (u_k)_m \overline{(u_k)_{n+1-m}}.$$

By the Cauchy–Schwarz inequality,

$$|s| \leq \sum_{k=1}^{r} \left| \sum_{m=1}^{n} (u_k)_m \overline{(u_k)_{n+1-m}} \right| \leq \sum_{k=1}^{r} \|u_k\| \|u_k^*\| = r < \sqrt{nr}.$$

Editor's Note. Most of our solvers noted that the case $n = 1$ had to be removed from the statement of the problem, otherwise $A = (1)$ is a counterexample. Eugen A. Herman proved the stronger inequality $|s| \leq \min(r, n - r + (1 - (-1)^n)/2)$.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robert Calcaterra, Con Amore Problem Group (Denmark), Eugen A. Herman, Omran Kouba (Syria), Tiberiu Trif (Romania), and the proposers.

Symmetrical triangles by reflected lines **October 2012**

1905. *Proposed by Luis Gonzales, Maracaibo, Venezuela, and Cosmin Pohoata, Princeton University, Princeton, NJ.*

Let ℓ be an arbitrary line in the plane of a given triangle ABC. The three lines obtained as the reflections of ℓ with respect to the sidelines of $\triangle ABC$ bound a triangle, namely $\triangle XYZ$.

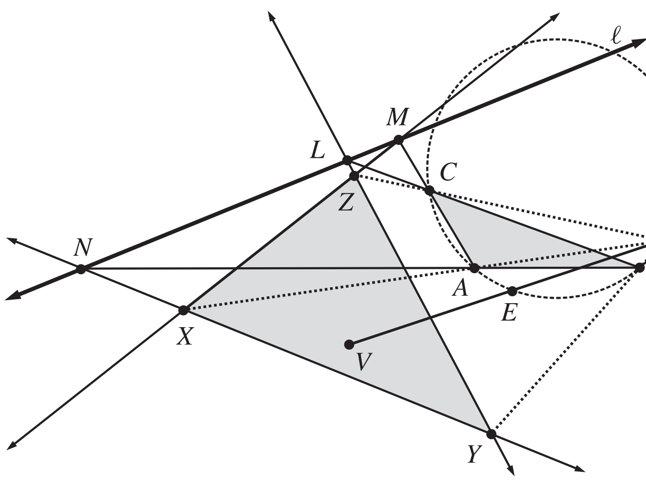
(a) Prove that the incenter of $\triangle XYZ$ lies on the circumcircle of $\triangle ABC$.

(b) If U and V are the incenter and the circumcenter of triangle XYZ, respectively, prove that the (second) intersection of the line UV with the circumcircle of $\triangle ABC$ is a fixed point E, independent of the position of ℓ. Moreover, prove that E is the Euler reflection point of $\triangle ABC$ (i.e., E is the intersection of the reflections of the Euler Line of $\triangle ABC$ with respect to its sidelines).

Solution by Michel Bataille, Rouen, France.
As stated, the problem is not correct. See the figure where the incenter of $\triangle XYZ$ is not on the circumcircle of $\triangle ABC$. We redefine U in order to obtain the problem that was

likely intended. We suppose that the lines YZ, ZX, and XY are the reflections of ℓ with respect to $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. We show that the lines AX, BY, and CZ are concurrent at a point U which is equidistant to the sidelines of $\triangle XYZ$ (therefore U is either the incenter or one of the excenters of $\triangle XYZ$). We assume that X, Y, and Z do not coincide (i.e., that the orthocenter H of $\triangle ABC$ is not on ℓ, as we will prove below).

Let L, M, and N be the intersections of ℓ with the lines BC, CA, and AB, respectively. It follows that the pairs of lines (YZ, BC), (ZX, CA), and (XY, AB) meet at L, M, and N, respectively. By Desargues' theorem, the lines AX, BY, and CZ meet at some point, say U. Point A is equidistant to ℓ and its reflection (line ZX) with respect to $\overline{AC}$. Point A is also equidistant to ℓ and its reflection (line XY) with respect to $\overline{AB}$. Therefore, A is equidistant to $\overline{ZX}$ and $\overline{XY}$. It follows that the line AX is one of the two bisectors of $\angle YXZ$. Similarly, the lines BY and CZ are bisectors of $\angle XYZ$ and $\angle YZX$, respectively, and so U is equidistant to the sidelines of $\triangle XYZ$.



We shall use complex numbers to prove parts (a) and (b). Let Γ be the circumcircle of $\triangle ABC$. Without loss of generality, we suppose that Γ is the unit circle centered at the origin. We will denote by the small letter p the complex affix of the point P (with this convention, $a\bar{a} = b\bar{b} = c\bar{c} = 1$ and $h = a + b + c$), and we will use w as a free complex variable. We may represent the line ℓ as the set of points whose affixes are of the form $w = h + t(\rho + \lambda i)$, where ρ and t are fixed numbers depending on ℓ with $\rho \in \mathbb{R}$, $|t| = 1$, and λ a variable real number. Since the line AB has equation $w + ab\bar{w} = a + b$, the affix w of a point of the reflection of ℓ with respect to $\overline{AB}$ is given by $h + t(\rho + \lambda i) + ab\bar{w} = a + b$. Taking conjugates and eliminating λ leads to the equation of the line $XY : t\bar{a}\bar{b}w + \bar{t}ab\bar{w} = -c\bar{t} - \bar{c}t - 2\rho$. Similarly, the equation of the line ZX is $t\bar{a}\bar{c}w + \bar{t}ac\bar{w} = -b\bar{t} - \bar{b}t - 2\rho$, and solving the system of these two equations, we obtain the affix of X: $x = u + 2\rho tu/(b + c)$ where $u = -abc/t^2$. In the same way, $y = u + 2\rho tu/(c + a)$ and $z = u + 2\rho tu/(a + b)$. Note that $x = y = z$ if and only if $\rho = 0$, that is, if and only if H is on ℓ. A simple calculation shows that $(x - u)/(u - a) = (\overline{x - u})/(\overline{u - a})$, so that A, X, and the point U with affix u are collinear. Similarly, the lines BY and CZ pass through U and thus point U is the one we defined before. Finally, note that $|u| = |abc|/|t|^2 = 1$, and thus U is on Γ. This completes the proof of (a).

The Euler line OH is obtained as the line ℓ passing through H (so that E is on Γ) and through O, which occurs when $\bar{h}t^2 = -h$. It follows that $e = abc \cdot \bar{h}/h =$

$(ab + bc + ca)/(a + b + c)$. Now, V is the point with affix

$$v = u + \frac{2\rho t u(ab + bc + ca)}{(a + b)(b + c)(c + a)}$$

because $|x - v| = |y - v| = |z - v| = 2|\rho|/(|a + b||b + c||c + a|)$, as it is readily checked. Moreover, an easy calculation shows that $(v - u)/(u - e) = \overline{(v - u)}/\overline{(u - e)}$. Hence U, V, and E are collinear points. This completes the proof of (b).

Editor's Note. Bruce S. Burdick observed that the problem as stated would be correct under the additional hypothesis of $\triangle ABC$ being acute. In this case, the point U defined as the intersection of the concurrent lines AX, BY, and CZ coincides with the incenter of $\triangle XYZ$.

Also solved by Bruce S. Burdick. There were three incomplete or incorrect solutions.

Answers

Solutions to the Quickies from page 289.

A1033. We place $ABCD$ in the complex plane. Because $ABCD$ is a simple quadrilateral, it follows that the midpoints of $\overline{BC}$ and $\overline{AD}$ do not coincide. Thus $B + C - A - D \neq 0$. If $P \in \{B, D\}$, then the (degenerate) triangles PAB and PCD are not similar because $A \neq C$ and $B \neq D$. If $P \notin \{B, D\}$, then the (possibly degenerate) triangles PAB and PCD are directly similar if and only if

$$\lambda = \frac{A - P}{B - P} = \frac{C - P}{D - P}.$$

Solving for P yields $P = (BC - AD)/(B + C - A - D)$. It follows that P exists and is unique. Finally, to exclude the degenerate case, note that the triples (A, B, P) and (C, D, P) are collinear if and only if $\lambda \in \mathbb{R}$. Because $\lambda = (A - C)/(B - D)$, it follows that (A, B, P) and (C, D, P) are collinear if and only if $\overline{AC}$ and $\overline{BD}$ are parallel, which can only happen if $ABCD$ is self-intersecting or collinear.

A1034. Our 2-digit printer's error is correct if and only if $(N, a, b) = (2^{2k-1}, 2, 2k)$ for $k \geq 2$, or $(N, a, b) = (a^{ka-1} - k, a, ka)$ for $a \geq 3$ and $k \geq 1$.

To prove this it is convenient to first get rid of some trivial cases. If $b \leq 1$, then $Na = a^b - b \leq a$, which is impossible since $N \geq 2$. If $a = 1$, then $b = 1 - N$ is likewise impossible. So we may assume that $2 \leq a < N$ and $2 \leq b < N$. Since $b \geq 1$ it follows that $a|a^b$, so $a|Na + b$, hence $a|b$. We may thus let $b = ka$ for some positive integer k. Thus $a^b = Na + b$ reduces to

$$a^{ka-1} = N + k.$$

If $a = 2$, then $N = 2^{2k-1} - k$. This is fine as long as $2^{2k-1} - k \geq 2$. For $k = 1$ this is false, but for $k \geq 2$,

$$N = 2^{2k-1} - k \geq 2k - k = k \geq 2,$$

where we have used the easily proven (and well known) fact that $2^m > m$ for all $m \geq 1$. This gives us our first set of solutions:

$$\left\{(N, a, b) = (2^{2k-1} - k, 2, 2k) : k \geq 2\right\}.$$

Finally, if $a \geq 3$ we have

$$N = a^{ka-1} - k > 2^{ka-1} - k \geq ka - k = k(a-1) \geq 2k \geq 2.$$

Our collection of solutions thus contains all of

$$\left\{(N, a, b) = (a^{ka-1} - k, a, ka) : a \geq 3 \text{ and } k \geq 1\right\},$$

and we have gathered all possibilities.

Notes. It is easy to check that there are no base-10 examples. The smallest possible base is 6, for which we have $2^4 = 24_6$. Dudeney's book mentioned in the statement of the problem is freely available online at Project Gutenberg:

http://www.gutenberg.org/files/16713/16713-h/16713-h.htm

REVIEWS

PAUL J. CAMPBELL, *Editor*
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Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Tao, Terence, Bounded gaps between primes (Polymath8)—a progress report, http://terrytao.wordpress.com/2013/06/30/bounded-gaps-between-primes-polymath8-a-progress-report/.

The twin prime conjecture is that there are infinitely many pairs of primes that differ by 2. If true, such a fact would be surprising, since primes tend to get rarer (their density decreases) as the integers increase, and because primes are about divisibility, not sums or differences. In May, Yitang Zhang (Univ. of New Hampshire) submitted a proof of a weaker claim, that there are infinitely many pairs of primes that differ by less than 70 million. That proof has been accepted, and the bound has already been reduced to 12,006—but can it be reduced to 2? Terence Tao says that the methods currently being explored cannot do it (maybe they could reach to 16); but stay tuned.

Kaznatcheev, Artem, Prime numbers: The 271 year old puzzle resolved, http://truthiscool.com/prime-numbers-the-271-year-old-puzzle-resolved.

In 1742, Christian Goldbach described in a letter to Euler his conjecture that every even integer larger than 2 can be written as the sum of 2 primes. A weaker version, that every odd integer larger than 5 can be written as the sum of 3 primes, appears to have been proved by Harald Andrés Helfgott (École Normale Supérieure–Paris). This weak ("ternary") Goldbach conjecture had been known to be true for all $n > e^{3100} \approx 2 \times 10^{1346}$ and for all $n < 4 \times 10^{18}$; and in 2012 Terence Tao (UCLA) had shown that any odd integer is the sum of at most 5 primes.

Stewart, Ian, The third culture: The power and glory of mathematics, http://www.newstatesman.com/sci-tech/2013/05/third-culture-power-and-glory-mathematics.

Famed mathematical expositor Ian Stewart begins by evoking C.P. Snow's earlier article in *New Statesman* and subsequent famous 1959 lecture about the chasm between the two differing cultures of the sciences and of the arts. Stewart claims that the gulf between the two has since narrowed "perceptibly," or at least that "the divide has been spanned by a number of bridges." He goes on to assert that mathematics partakes of each of the cultures but is not contained in either (nor "in their union"): "not entirely a science, and even less plausibly an art." He finds the funding of mathematics to be "pathetic" in comparison to that of "particle physics, radio astronomy, and genetics." But his suggestion of greater funding for supercomputers will tend to reinforce public misunderstanding that mathematics is computing, and his remarks further on could be taken as characterizing statistics as mathematics. Stewart cites the value of routine uses of mathematics (though "to be useful … it has to be invisible") but also claims for the subject an "intrinsic intellectual interest" with its own demands and opportunities for creativity. Perhaps this apologia for mathematics will be convincing for its intended general audience, and I doubt that I could do as well; I just wish that it sounded more convincing to me.

Math. Mag. **86** (2013) 296–297. doi:10.4169/math.mag.86.4.296. © Mathematical Association of America

Frenkel, Edward, *Love and Math: The Heart of Hidden Reality*, Basic Books, 2013; ii + 300 pp, $27.99 (P). ISBN 978-0-465-05074-1.

I find the title of this book awkward, even mawkish; yet author Frenkel certainly has a love affair with mathematics. He describes in interesting fashion his academic history, including transcending obstacles (such as anti-Semitism in Russia in the 1980s) in the pursuit of his love of mathematics. A singular achievement of the book is to explore the nature of mathematics through several levels of nontechnical description of—of all things!—the Langlands Program! That research program, announced by Robert Langlands in 1967, conjectures a relationship between automorphic functions and representations of the Galois group of symmetries of a number field, with corresponding relationships also in the world of curves over finite fields and that of Riemann surfaces. (Now I appreciate Kac-Moody algebras.) The exposition deftly avoids getting bogged down in too many details. Frenkel also uses symmetry as a touchstone for describing the nature of mathematics as "the study of abstract objects and concepts," whose basic characteristics are universality, objectivity, endurance, and relevance to the physical world. He asserts that the usefulness of mathematicians to others is the "mathematical mindset": the "ability to formulate the right questions and then go through a cold and unbiased analysis to get the answers." A final chapter takes the title in what to me (but not to Frenkel) is a totally different direction, as he details the development of his 2010 short film *Rites of Love and Math*, a major feature of which is tattooing a mathematical formula of his on a woman's belly.

Wells, David, *Games and Mathematics: Subtle Connections*, Cambridge University Press, 2012; x + 246 pp, $80, $19.99 (P). ISBN 978-1-107-02460-1, 978-1-107-69091-2.

How are mathematical recreations different from everyday puzzles? Why and how is chess not mathematics? And what aspects of chess *are* mathematical? This perceptive book offers splendid answers amid a multitude of examples. Author Wells attributes the popularity of puzzles to their challenge and mystery, but characterizes mathematics in terms of an inseparable triad of game-playing, science, and "observing" (mathematicians "spot patterns and connections, observe analogies, notice unusual sequences of moves, . . . and generalise their results, leading to ideas of structure"). Even the general reader can learn a lot about the nature of mathematics from this book (despite some use of differentiation without notice or apology), but there is special advice for students: "Students who do not 'play around' with . . . new ideas, but stick to the textbook explanation and do no more than answer a few exercises, will never develop deep intuition and never become real mathematicians."

Procaccia, Ariel D., Cake cutting: not just child's play, *Communications of the Association for Computing Machinery* 56 (7) (July 2013) 78–87.

This article focuses on "fair allocation of divisible goods," a topic in social welfare theory that has been addressed by economists and mathematicians. Major concepts are proportionality and envy-freeness; but since the audience for this article is computer scientists, the emphasis is on the computational complexity of allocation problems and the efficiency of algorithms for them (a result by the Editor of this MAGAZINE is cited). Also included are the game-theoretic approach (which seeks strategy-proof algorithms) and optimality from society's point of view (in terms of the sum of the players' utilities). Finally, author Procaccia points out how cake-cutting can be applied to problems in computer science.

Posamentier, Alfred S., and Ingmar Lehmann, *Magnificent Mistakes in Mathematics*, Prometheus Books, 2013; 298 pp, $24. ISBN 978-1-61614-747-1.

Klymchuk, Sergiy, and Susan Staples, *Paradoxes and Sophisms in Mathematics*, MAA, 2013; xiii + 98 pp, $40 (P) ($32 to MAA members). ISBN 978-0-88385-781-6.

Both of these books feature mathematical results that are just not right (e.g., $0 = 1$). Posamentier and Lehmann list mistakes in arithmetic, algebra, geometry, and probability and statistics, including some by famous mathematicians. Klymchuk and Staples offer paradoxes and mistakes that arise in functions, limits, derivatives, and integrals; half of their book is devoted to posing the problems, with solutions in the second half. (The MAA recently restructured its membership categories and fees; perhaps it should also consider revising the costs of its books, since even the member price for *Paradoxes and Sophisms* seems high.)

NEWS AND LETTERS

42nd USA Mathematical Olympiad
4th USA Junior Mathematical Olympiad

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This year the Committee on the American Mathematics Competitions offered the USA Junior Mathematical Olympiad (USAJMO) for the fourth time for students in 10th grade and below. Our experience of the last three years shows that it provides a nicely balanced link between the computational character of the AIME problems and the proof-oriented problems of the USAMO. This year the competitions took place on April 30 and May 1. The USA Junior Mathematical Olympiad contained three problems for each of two days, with an allowed time of 4.5 hours each day—the same as the USAMO. Problems JMO1, JMO2, JMO4, and JMO5 were different from the USAMO problems, but JMO3 and JMO6 were the same as USAMO1 and USAMO4, respectively.

USAMO Problems

1. In triangle ABC, points P, Q, R lie on sides BC, CA, AB, respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z respectively. Prove that $YX/XZ = BP/PC$.
2. For a positive integer $n \geq 3$ plot n equally spaced points around a circle. Label one of them A, and place a marker at A. One may move the marker forward in a clockwise direction to either the next point or the point after that. Hence there are a total of $2n$ distinct moves available; two from each point. Let a_n count the number of ways to advance around the circle exactly twice, beginning and ending at A, without repeating a move. Prove that $a_{n-1} + a_n = 2^n$ for all $n \geq 4$.
3. Let n be a positive integer. There are $n(n+1)/2$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An *operation* is to choose a line parallel to one of the sides of the triangle, and flipping all the marks on that line. A configuration is called *admissible* if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

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4. Find all real numbers $x, y, z \geq 1$ satisfying

$$\min\left(\sqrt{x+xyz}, \sqrt{y+xyz}, \sqrt{z+xyz}\right) = \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

5. Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.
6. Let ABC be a triangle. Find all points P on segment BC satisfying the following property: If X and Y are the intersections of line PA with the common external tangent lines of the circumcircles of triangles PAB and PAC, then

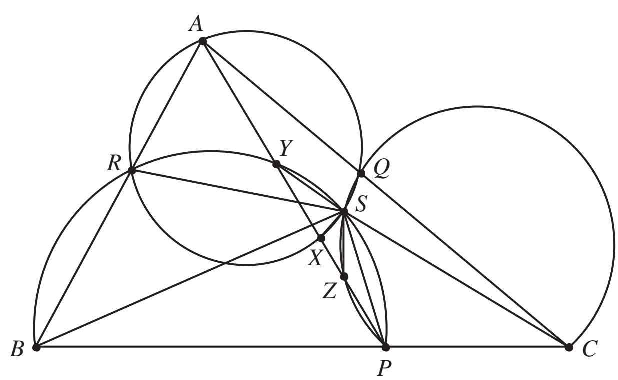
$$\left(\frac{PA}{XY}\right)^2 + \frac{PB \cdot PC}{AB \cdot AC} = 1.$$

Solutions

1. Assume that ω_B and ω_C intersect again at another point S (other than P). (The degenerate case of ω_B and ω_C being tangent at P can be dealt similarly.) Because $BPSR$ and $CPSQ$ are cyclic, we have $\angle RSP = 180^\circ - \angle PBR$ and $\angle PSQ = 180^\circ - \angle QCP$. Hence, we obtain

$$\begin{aligned}\angle QSR = 360^\circ - \angle RSP - \angle PSQ &= \angle PBR + \angle QCP \\ &= \angle CBA + \angle ACB = 180^\circ - \angle BAC,\end{aligned}$$

from which it follows that $ARSQ$ is cyclic; that is, $\omega_A, \omega_B, \omega_C$ meet at S. (This is Miquel's theorem.)



Because $BPSY$ is inscribed in ω_B, $\angle XYS = \angle PYS = \angle PBS$. Because $ARXS$ is inscribed in ω_A, $\angle SXY = \angle SXA = \angle SRA$. Because $BPSR$ is inscribed in ω_B, $\angle SRA = \angle SPB$. Thus, we have $\angle SXY = \angle SRA = \angle SPB$. In triangles SYX and SBP, we have $\angle XYS = \angle PBS$ and $\angle SXY = \angle SPB$. Therefore, triangles SYX and SBP are similar to each other, and, in particular,

$$\frac{YX}{BP} = \frac{SX}{SP}.$$

Similarly, we can show that triangles SXZ and SPC are similar to each other and that

$$\frac{SX}{SP} = \frac{XZ}{PC}.$$

Combining the last two equations yields the desired result.

This problem and solution were suggested by Zuming Feng.

2. We will show that $a_n = \frac{1}{3}(2^{n+1} + (-1)^n)$. This is sufficient, since then we would have

$$a_{n-1} + a_n = \tfrac{1}{3}(2^n + (-1)^{n-1}) + \tfrac{1}{3}(2^{n+1} + (-1)^n) = \tfrac{1}{3}\,(2^n + 2 \cdot 2^n) = 2^n.$$

We will need the fact that for all positive integers n,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^k = \tfrac{1}{3}(2^{n+1} + (-1)^n).$$

This may be established by strong induction. To begin, the cases $n = 1$ and $n = 2$ are quickly verified. Now suppose that $n \geq 3$ is odd, say $n = 2m + 1$. We find that

$$\begin{aligned}\sum_{k=0}^{m} \binom{2m+1-k}{k} 2^k &= 1 + \sum_{k=1}^{m} \binom{2m-k}{k} 2^k + \sum_{k=1}^{m} \binom{2m-k}{k-1} 2^k \\ &= \sum_{k=0}^{m} \binom{2m-k}{k} 2^k + 2 \sum_{k=0}^{m-1} \binom{2m-1-k}{k} 2^k \\ &= \tfrac{1}{3}(2^{2m+1} + 1) + \tfrac{2}{3}(2^{2m} - 1) \\ &= \tfrac{1}{3}(2^{2m+2} - 1),\end{aligned}$$

using the induction hypothesis for $n = 2m$ and $n = 2m - 1$. For even n the computation is similar, so we omit the steps. This proves the claim.

We now determine the number of ways to advance around the circle twice, organizing our count according to the points visited both times around the circle. It is straightforward to check that no two such points may be adjacent, and that there are exactly two sequences of moves leading from any such point to the next. (These sequences involve only moves of length two except possibly at the endpoints.) Hence given $k \geq 1$ points around the circle, no two adjacent and not including point A, there would appear to be 2^k ways to traverse the circle twice without repeating a move. However, half of these options lead to repeating the same route twice, giving 2^{k-1} ways in actuality. There are $\binom{n-k}{k}$ ways to select k nonadjacent points on the circle not including A (add an extra point behind each of k chosen points), for a total contribution of

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{k-1} = \frac{1}{2}\left[-1 + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^k\right] = \frac{1}{6}(2^{n+1} + (-1)^n) - \frac{1}{2}.$$

On the other hand, if the $k \geq 1$ nonadjacent points do include point A then there are $\binom{n-k-1}{k-1}$ ways to choose them around the circle. (Select A but not the next point, then add an extra point after each of $k - 1$ selected points.) But now there are actually 2^k ways to circle twice, since we can choose either move at A and the subsequent points, then select the other options the second time around. Hence the contribution in this case is

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} 2^k = 2 \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-2-k}{k} 2^k = \frac{2}{3}(2^{n-1} + (-1)^n).$$

Finally, if n is odd then there is one additional way to circle in which no point is visited twice by using only steps of length two, giving a contribution of

$\frac{1}{2}(1-(-1)^n)$. Therefore the total number of paths is

$$\tfrac{1}{6}(2^{n+1}+(-1)^n)-\tfrac{1}{2}+\tfrac{2}{3}(2^{n-1}+(-1)^n)+\tfrac{1}{2}(1-(-1)^n),$$

which simplifies to $\frac{1}{3}(2^{n+1}+(-1)^n)$, as desired.

This problem and solution were suggested by Sam Vandervelde.

3. For $n=1$ the answer is clearly 1, since there is only one configuration other than the initial one, and that configuration takes one step to get to. From now on we will consider $n \geq 2$.

Note that there are $3n$ possible operations in total, since we can select $3n$ lines to perform an operation on (n lines parallel to each side of the triangle.) Performing an operation twice on the same line is equivalent to doing nothing. Hence, we will describe any combination of operations as a triple of n-tuples $((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n))$, where each element a_i, b_i, c_i is either 0 or 1 (0 means no operation, 1 means the opposite); each tuple of the triple denotes operating on a line parallel to one of the sides; and the indices, i.e., $1, 2, \ldots, n$, denote the number of marks in the row of operation. Let A denote the set of all such $3n$-tuples. Hence $|A| = 2^{3n}$.

Let B denote the set of all admissible configurations. Let $N = n(n+1)/2$. We will describe each element of B by an N-tuple $(z_1, z_2, \ldots, z_N)$, where each element is either 0 or 1 (0 means black, 1 means white). (Which element refers to which position is not important.)

For each element $a \in A$, let $b = f(a)$ be the element of B that is the result of applying the operations in a. Then $f(a+a') = f(a) + f(a')$ for all $a, a' \in A$, where addition is considered in modulo 2. Let K be the set of all $a \in A$ such that $f(a)$ is the all-black configuration. The following eight elements are easily seen to be in K.

- $((0,0,\ldots,0),(0,0,\ldots,0),(0,0,\ldots,0)) = \mathrm{id}$
- $((0,0,\ldots,0),(1,1,\ldots,1),(1,1,\ldots,1)) = x$
- $((1,1,\ldots,1),(1,1,\ldots,1),(0,0,\ldots,0)) = y$
- $((1,1,\ldots,1),(0,0,\ldots,0),(1,1,\ldots,1)) = x+y$
- $((0,1,0,1,\ldots),(0,1,0,1,\ldots),(0,1,0,1,\ldots)) = z$
- $((0,1,0,1,\ldots),(1,0,1,0,\ldots),(1,0,1,0,\ldots)) = x+z$
- $((1,0,1,0,\ldots),(1,0,1,0,\ldots),(0,1,0,1,\ldots)) = y+z$
- $((1,0,1,0,\ldots),(0,1,0,1,\ldots),(1,0,1,0,\ldots)) = x+y+z$

We will show that they are the only elements of K.

Suppose that $L = ((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n), (c_1, c_2, \ldots, c_n))$ is in K. Then $a_i + b_j + c_k = 0$ whenever $i+j+k = 2n+1$ (why is left as an exercise for the reader). By adding x and/or y if necessary, we will assume that $b_n = c_n = 0$. Since $a_2 + b_{n-1} + c_n = a_2 + b_n + c_{n-1} = 0$, we have that $b_{n-1} = c_{n-1}$. There are two cases:

(a) $b_{n-1} = c_{n-1} = 0$. Then from $a_3 + b_{n-2} + c_n = a_3 + b_{n-1} + c_{n-1} = a_3 + b_n + c_{n-2}$, we have that $b_{n-2} = c_{n-2} = 0$. Continuing in this manner (considering equalities with $a_4, a_5, \ldots$), we find that all the b_i's and c_i's are 0, from which we deduce that $L = \mathrm{id}$.

(b) $b_{n-1} = c_{n-1} = 1$. Then from $a_3 + b_{n-2} + c_n = a_3 + b_{n-1} + c_{n-1} = a_3 + b_n + c_{n-2}$, we have that $b_{n-2} = c_{n-2} = 0$. Continuing in this manner (considering equalities with $a_4, a_5, \ldots$), we find that $(b_1, b_2, \ldots, b_n) = (c_1, c_2, \ldots, c_n) = (\ldots, 1, 0, 1, 0)$, from which we deduce that either $L = z$ or $L = x + z$.

Hence L is one of the eight elements listed above. It follows that the 2^{3n} elements of A form 2^{3n-3} sets, each set corresponding to an element of B. For each element $a \in A$, let x_1 be the number of $a_1, a_3, \ldots$ that are 1, and let x_2 be the number of $a_2, a_4, \ldots$ that are 1. Define y_1, y_2, z_1, and z_2 similarly with the b_i's and c_i's. We want to find the element in the set containing a that has the smallest value of $T = x_1 + x_2 + y_1 + y_2 + z_1 + z_2$. The maximum of this value over all the sets is the desired answer.

We observe that an element $a \in A$ has the minimal value of T in its set if and only if it satisfies the following inequalities:

(a) $x_1 + x_2 + y_1 + y_2 \leq n$

(b) $x_1 + x_2 + z_1 + z_2 \leq n$

(c) $y_1 + y_2 + z_1 + z_2 \leq n$

(d) $x_2 + y_2 + z_2 \leq \left\lfloor \dfrac{3\lfloor n/2 \rfloor}{2} \right\rfloor = V$

(e) $x_1 + y_1 + z_2 \leq \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

(f) $x_2 + y_1 + z_1 \leq \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

(g) $x_1 + y_2 + z_1 \leq \left\lfloor \dfrac{2\lceil n/2 \rceil + \lfloor n/2 \rfloor}{2} \right\rfloor = W$

We wish to find the maximal value of T that an element satisfying all these inequalities can have. Adding the last four inequalities and dividing by 4, we obtain $T \leq \lfloor (V + 3W)/2 \rfloor$. We consider four cases:

(a) $n = 4k$. $V = W = 3k$, and so $T \leq 6k$. We can choose $x_1 = x_2 = y_1 = y_2 = z_1 = z_2 = k$ to attain the bound.

(b) $n = 4k + 1$. $V = 3k$ and $W = 3k + 1$, and so $T \leq 6k + 1$. We can choose $x_1 = x_2 = y_1 = y_2 = z_2 = k$ and $z_1 = k + 1$ to attain the bound.

(c) $n = 4k + 2$. $V = 3k + 1$ and $W = 3k + 1$, and so $T \leq 6k + 2$. We can choose $x_1 = x_2 = y_1 = y_2 = k$ and $z_1 = z_2 = k + 1$ to attain the bound.

(d) $n = 4k + 3$. $V = 3k + 1$ and $W = 3k + 2$, and so $T \leq 6k + 3$. We can choose $x_1 = x_2 = y_2 = k$ and $y_1 = z_1 = z_2 = k + 1$ to attain the bound.

This concludes our proof.

This problem and solution were suggested by Warut Suksompong.

4. Let a, b, c be nonnegative real numbers such that $x = 1 + a^2$, $y = 1 + b^2$ and $z = 1 + c^2$. We may assume that $c \leq a, b$, so that the condition of the problem becomes

$$(1 + c^2)(1 + (1 + a^2)(1 + b^2)) = (a + b + c)^2.$$

The Cauchy–Schwarz inequality yields

$$(a + b + c)^2 \leq (1 + (a + b)^2)(c^2 + 1).$$

Combined with the previous relation, this shows that

$$(1 + a^2)(1 + b^2) \leq (a + b)^2,$$

which can also be written $(ab - 1)^2 \leq 0$. Hence $ab = 1$ and the Cauchy–Schwarz inequality must be an equality, that is, $c(a + b) = 1$. Conversely, if $ab = 1$ and $c(a + b) = 1$, then the relation in the statement of the problem holds, since $c = 1/(a + b) < 1/b = a$, and similarly $c < b$.

Thus the solutions of the problem are

$$x = 1 + a^2, \quad y = 1 + \frac{1}{a^2}, \quad z = 1 + \left(\frac{a}{a^2+1}\right)^2,$$

for some $a > 0$, as well as permutations of this. (Note that we can actually assume $a \geq 1$ by switching x and y if necessary.)

This problem and solution were suggested by Titu Andreescu.

5. For a given positive integer k, write $10^k m - n = 2^r 5^s t$, where $\gcd(t, 10) = 1$. For large enough values of k, the number of times 2 and 5 divide the left-hand side is at most the number of times they divide n; hence by choosing k large we can make t arbitrarily large. Choose k so that t is larger than either m or n.

   Since t is relatively prime to 10 there is a smallest exponent b for which $t \mid (10^b - 1)$. Thus b is the number of digits in the repeating portion of the decimal expansion for $1/t$. More precisely, if we write $tc = (10^b - 1)$, then the repeating block is the b-digit decimal representation of c, obtained by prepending extra initial zeros to c as necessary. Since t is larger than m or n, the decimal expansions of m/t and n/t will consist of repeated b-digit representations of cm and cn, respectively. Rewriting the identity in the first line as

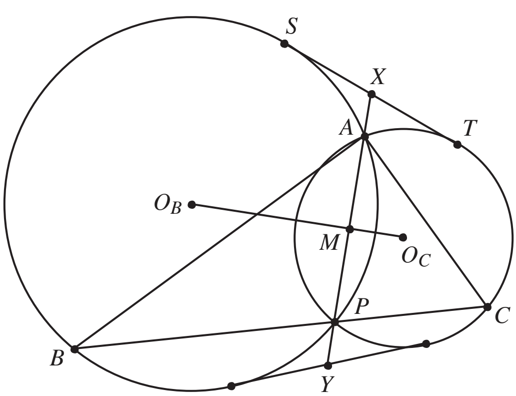
$$10^k\left(\frac{m}{t}\right) = 2^r 5^s + \frac{n}{t},$$

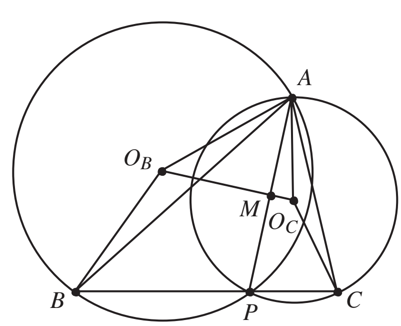
   we see that the decimal expansion of n/t is obtained from that of m/t by shifting the decimal to the right k places and removing the integer part. Thus the b-digit representations of cm and cn are cyclic shifts of one another. In particular, they have the same number of occurrences of each nonzero digit. (Because they may have different numbers of leading zeros as b-digit numbers, the number of zeros in their decimal expansions may differ.)

   This problem and solution were suggested by Richard Stong.

6. We consider the left-hand configuration below. Let O_B and ω_B (O_C and ω_C) denote the circumcenter and circumcircle of triangle ABP (ACP) respectively. Line ST, with S on ω_B and T on ω_C, is one of the common tangent lines of the two circumcircles. Point X lies on segment ST. Point Y lies on the other common tangent line.

   We start with the following simple and well-known facts of geometry.





Let M be the intersection of segments XY and O_BO_C. By symmetry, M is the midpoint of both segments AP and XY, and line O_BO_C is the perpendicular bisector of segments XY and AP. By the power-of-a-point theorem,

$$XS^2 = XA \cdot XP = XT^2 \quad \text{and} \quad X \text{ is the midpoint of segment } ST. \tag{1}$$

Triangles ABC and AO_BO_C are similar to each other, which is the so-called *Salmon theorem*. Indeed, $\angle ABC = \angle MO_BA = \angle O_CO_BA$, because each angle is equal to half of the angular size of arc AP of ω_B. Likewise, $\angle O_BO_CA = \angle BCA$. In particular, we have

$$\frac{AB}{AO_B} = \frac{BC}{O_BO_C} = \frac{CA}{O_CA}. \tag{2}$$

Set $AB = c$, $BC = a$, and $CA = b$. We will establish the following key fact:

$$1 - \left(\frac{PA}{XY}\right)^2 = \frac{BC^2}{(AB+AC)^2} = \frac{a^2}{(b+c)^2}. \tag{3}$$

With this fact, the given condition in the problem becomes

$$\frac{PB \cdot PC}{AB \cdot AC} = \frac{a^2}{(b+c)^2} \quad \text{or} \quad PB \cdot PC = \frac{a^2bc}{(b+c)^2}. \tag{4}$$

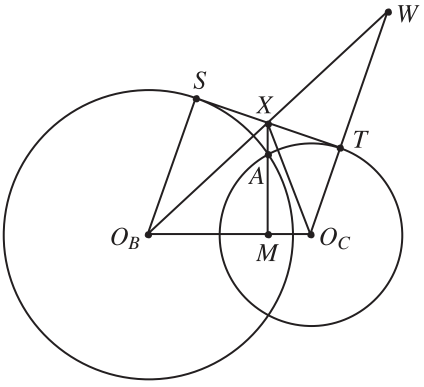
There are precisely two points P_1 and P_2 (on segment BC) satisfying (4): AP_1 is the bisector of $\angle BAC$ and P_2 is the reflection of P_1 across the midpoint of segment BC. Indeed, by the angle-bisector theorem, $P_2C = P_1B = ac/(b+c)$ and $P_2B = P_1C = ab/(b+c)$, from which (4) follows.

In order to settle the question, it remains to show that we can't have more than two points satisfying (4). We just write (4) as

$$\frac{a^2bc}{(b+c)^2} = PB \cdot PC = PB \cdot (a - PB).$$

This a quadratic equation in PB, which can have at most two solutions.

Now we establish the key fact (3).



By the power-of-a-point theorem, we have $XA \cdot XP = XS^2$. Therefore,

$$1 - \left(\frac{PA}{XY}\right)^2 = \frac{XY^2 - PA^2}{XY^2} = \frac{(XY+PA)(XY-PA)}{XY^2}$$

$$= \frac{4XA \cdot XP}{XY^2} = \frac{4XS^2}{XY^2} = \frac{ST^2}{XY^2}. \tag{5}$$

Let S_1 and T_1 be the feet of the perpendiculars from S and T to line O_BO_C. It is easy to see that right triangles O_BSS_1, O_CTT_1, O_SO_CU are similar to each other. Note also that XM is the midline of right trapezoid S_1STT_1 (because of (1)). Therefore, we have

$$\frac{ST}{O_BO_C} = \frac{UO_C}{O_BO_C} = \frac{S_1S}{O_BS} = \frac{T_1T}{O_CT} = \frac{S_1S + T_1T}{O_BS + O_CT} = \frac{2XM}{O_BS + O_CT} = \frac{XY}{O_BS + O_CT},$$

or, by (2),

$$\frac{ST}{XY} = \frac{O_BO_C}{O_BS + O_CT} = \frac{O_BO_C}{O_BA + O_CA} = \frac{BC}{BA + CA} = \frac{a}{b+c}. \qquad (6)$$

It is clear that (3) follows from (5) and (6).

This problem and solution were suggested by Titu Andreescu and Cosmin Pohoata. Zuming Feng suggested a simplified derivation of (3).

USAJMO Problems

1. Are there integers a and b such that $a^5b + 3$ and $ab^5 + 3$ are both perfect cubes of integers?
2. Each cell of an $m \times n$ board is filled with some nonnegative integer. Two numbers in the filling are said to be *adjacent* if their cells share a common side. (Note that two numbers in cells that share only a corner are not adjacent.) The filling is called a *garden* if it satisfies the following two conditions:
   (i) The difference between any two adjacent numbers is either 0 or 1.
   (ii) If a number is less than or equal to all of its adjacent numbers, then it is equal to 0.

   Determine the number of distinct gardens in terms of m and n.
3. Same as USAMO 1.
4. Let $f(n)$ be the number of ways to write n as a sum of powers of 2, where we keep track of the order of the summation. For example, $f(4) = 6$ because 4 can be written as 4, $2+2$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Find the smallest n greater than 2013 for which $f(n)$ is odd.
5. Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter XY. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line XY. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that
$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$
6. Same as USAMO 4.

Solutions

1. The answer is negative. Modulo 9, a cube is 0 or ± 1. Assuming that one of $a^5b + 3$ and $ab^5 + 3$ is 0 mod 9, it follows that at least one of the numbers a and b, say a, is divisible by 3, hence $a^5b + 3$ is 3 mod 27, not a perfect cube. If $a^5b + 3$ and $ab^5 + 3$ are both perfect cubes of the form ± 1 mod 9, then a^5b and ab^5 are both

7 or 5 mod 9, and so their product $(ab)^6$ is -1, -2, or 4 mod 9. But $(ab)^6$ is the square of a perfect cube not divisible by 3, so is precisely 1 mod 9, a contradiction.

This problem and solution were suggested by Titu Andreescu.

2. First note that if $m = n = 1$, then condition (ii) is satisfied, so the one cell must contain 0. Henceforth, we assume that $m > 1$ or $n > 1$, so that every cell has at least one adjacent cell.

   We define the distance between two cells to be $|x_1 - x_2| + |y_1 - y_2|$, where (x_1, y_1), (x_2, y_2) are the centers of the respective cells. In particular, two cells are adjacent if and only if the distance between them is 1.

   By condition (ii), the minimum value in a cell of a garden is 0. In particular, a garden has at least one zero.

   Suppose that a cell in a garden contains an integer $k \geq 1$. By condition (ii), it has an adjacent cell with a smaller integer. Since the difference is either 0 or 1, it must be 1. Thus, a cell assigned k will have an adjacent cell assigned $k - 1$. We draw a line segment between the two center points of these two cells. Repeating this procedure, we can find a path from k to a 0-cell. We call such a path a *garden path.* There may be more than one garden path from a given cell, but all such paths will have length k.

   Suppose that for some cell C assigned k, there is a path from C to a 0-cell D such that the distance from C to D is less than k. Take an arbitrary path from C to D, and let the numbers in the cells the path goes through be $a_0 = k, a_1, \ldots, a_n = 0$. Now $a_i - a_{i+1} \leq 1$, so

$$k = \sum_{i=0}^{n-1} (a_i - a_{i+1}) \leq n < k,$$

   a contradiction. Thus, the nearest 0-cell to C has distance $\geq k$ from C. By the previous paragraph, there exists a path from C to a 0-cell with distance k. Therefore, the distance to the nearest 0-cell is k.

   Therefore, the assignment of non-zero numbers is uniquely determined by the position of zeros. Consequently, we just need to count the number of ways to put zeros in mn cells, subject to the condition that there is at least one zero. This is clearly $2^{mn} - 1$.

   This problem and solution were suggested by Sungyoon Kim.

3. Same as USAMO 1.

4. The answer is 2047. We show that $f(n)$ is odd if and only if n is of the form $2^k - 1$.

   We use the method of generating functions. Define the formal power series $b(x) = \sum_{j=0}^{\infty} x^{2^j}$. The desired statement can be interpreted as

$$1/(1 - b(x)) \equiv b(x)/x \pmod 2,$$

   where the congruence means that the difference between the two sides has all coefficients divisible by 2. It is equivalent to prove the same thing after clearing denominators, in other words,

$$b(x)^2 - b(x) \equiv x \pmod 2.$$

   But this holds because $b(x)^2 \equiv b(x^2) \pmod 2$ (all the cross terms in the expansion of $b(x)^2$ being even), so

$$b(x)^2 - b(x) \equiv b(x^2) - b(x) \equiv x \pmod 2.$$

   This problem and solution were suggested by Kiran Kedlaya and David Speyer.

5. Note that $\angle XAY = \angle XBY = \angle XCY = \angle PZX = \angle PZY = 90°$. In right triangles BXY, AXY, AXP, we have

$$BY = XY \cos\angle BYX,$$
$$AX = XY \cos\angle AXY,$$
$$XP = \frac{AX}{\cos\angle AXP} = \frac{XY\cos\angle AXY}{\cos\angle AXP},$$

from which it follows that

$$\frac{BY}{XP} = \frac{\cos\angle BYX \cos\angle AXP}{\cos\angle AXY}.$$

Likewise, we have

$$\frac{CY}{XQ} = \frac{\cos\angle CYX \cos\angle AXQ}{\cos\angle AXY}.$$

Adding the last two equations yields

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos\angle BYX\cos\angle AXP + \cos\angle CYX\cos\angle AXQ}{\cos\angle AXY}. \tag{7}$$

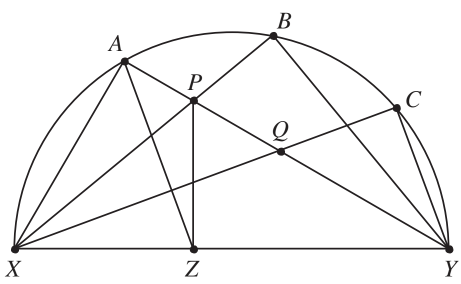
Because both CY and AZ are perpendicular to XC, $\angle CYX = \angle AZX$. Because $\angle XAP = \angle XZP = 90°$, quadrilateral $AXZP$ is cyclic, from which it follows that $\angle AZX = \angle APX$. Therefore, we have $\angle CYX = \angle AZX = \angle APX = 90° - \angle AXP$ or $\angle CYX + \angle AXP = 90°$. Likewise, we can show that $\angle BYX + \angle AXQ = 90°$. Consequently, we conclude that $\cos\angle BYX = \sin\angle AXQ$ and $\sin\angle CYX = \cos\angle AXP$. Thus, by the addition and subtraction formula, (7) becomes

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\sin\angle AXQ \sin\angle CYX + \cos\angle CYX\cos\angle AXQ}{\cos\angle AXY}$$
$$= \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY}.$$

Because $ACYX$ is cyclic, $\angle AXQ = \angle AXC = \angle CYA$, implying that $\angle CYX - \angle AXQ = \angle CYX - \angle CYA = \angle AYX$. Therefore,

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{\cos(\angle CYX - \angle AXQ)}{\cos\angle AXY}$$
$$= \frac{\cos\angle AYX}{\cos\angle AXY} = \frac{\sin\angle AXY}{\cos\angle AXY} = \tan\angle AXY = \frac{AY}{AX},$$

as desired.



This problem and solution were suggested by Zuming Feng.

6. Same as USAMO 4.

The top twelve students on the 2013 USAMO were (in alphabetical order):

Calvin Deng	12	North Carolina School Science/Math	NC
Andrew He	10	Monta Vista HS	CA
Ravi Jagadeesan	11	Phillips Exeter Academy	NH
Pakawut Jiradilok	13	Phillips Exeter Academy	NH
Ray Li	12	Phillips Exeter Academy	NH
Kevin Li	10	West Windsor-Plainsboro HS South	NJ
Mark Sellke	11	William Henry Harrison HS	IN
Bobby Shen	12	Dulles HS	TX
Zhuoqun Song	10	Phillis Exeter Academy	NH
David Stoner	10	South Aiken HS	SC
Thomas Swayze	12	Canyon Crest Academy	CA
Victor Wang	12	Ladue Horton Watkins HS	MO

The top eleven students on the 2013 USAJMO were (in alphabetical order):

Ryan Alweiss	10	(AAST) Bergen County Academies	NJ
Artur Dennis	9	(AAST) Bergen County Academies	NJ
Brian Gu	9	Interlake	WA
Brice Huang	10	West Windsor-Plainsboro HS North	NJ
Shashwat Kishore	10	Unionville HS	PA
Max Murin	10	Lakeside School	WA
Saranesh Prembabu	10	Dougherty Valley HS	CA
Michael Seaman	10	Monmouth University (Seaman Homeschool)	NJ
Sean Shi	9	Saratoga HS	CA
Kevin Wang	10	(AAST) Besrgen County Academies	NJ
KuoAn Wei	10	Phillips Exeter Academy	NH

54th International Mathematical Olympiad

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Problems (Day 1)

1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\cdots\left(1 + \frac{1}{m_k}\right).$$

2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

   - no line passes through any point of the configuration;
   - no region contains points of both colours.

   Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

3. Let the excircle of the triangle ABC opposite the vertex A be tangent to side BC at A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

   The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.

Problems (Day 2)

4. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y, and H are collinear.

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5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:
   (i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;
   (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;
   (iii) there exists a rational number $a > 1$ such that $f(a) = a$.

   Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.
6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labeled a and d does not intersect the chord joining the points labeled b and c.

   Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

Solutions

1. We induct on k. The base case $k = 1$ holds with $m_1 = n$. If the statement holds for some k, we consider two cases. If $n = 2m - 1$ is odd, we have

$$1 + \frac{2^{k+1}-1}{n} = \frac{2m}{2m-1}\frac{2^{k+1}+2m-2}{2m} = \left(1 + \frac{1}{2m-1}\right)\left(1 + \frac{2^k-1}{m}\right),$$

where $1 + \frac{2^k-1}{m}$ is the product of k terms of the desired form by the induction hypothesis, yielding the desired decomposition. If $n = 2m$ is even, we have

$$\begin{aligned} 1 + \frac{2^{k+1}-1}{n} &= \frac{2^{k+1}+2m-1}{2^{k+1}+2m-2}\ \frac{2^{k+1}+2m-2}{2m} \\ &= \left(1 + \frac{1}{2^{k+1}+2m-2}\right)\left(1 + \frac{2^k-1}{m}\right), \end{aligned}$$

where $1 + \frac{2^k-1}{m}$ is the product of k terms of the desired form by the induction hypothesis, yielding the desired decomposition and completing the induction.

   This problem was proposed by the Olympiad problem committee from Japan.
2. The answer is 2013. We first show that a good arrangement with at most 2013 lines always exists. We begin with the following key lemma.

   LEMMA 1. *Any pair of points P and Q in a Colombian configuration C can be separated from the other points by two lines.*

   Proof. No three points in C are collinear, so each other point has one of finitely many positive distances to PQ. Choose $r > 0$ less than all such distances; the two lines parallel to and at a distance r from PQ have the desired property. ■

   Let $\mathcal{C}$ be the convex hull of our Colombian configuration C. If a red point R is a vertex of $\mathcal{C}$, draw a line ℓ_1 separating R from all other points. Next, place the other 2012 red points into 1006 pairs and apply Lemma 1 to draw 2012 lines separating them from the rest of C. Together with ℓ_1, these 2012 lines form a good arrangement. Otherwise, $\mathcal{C}$ has a side B_1B_2 consisting of blue points. Draw a line ℓ_1 separating B_1 and B_2 from the rest of C, place the 2012 remaining blue points into

1006 pairs, and apply Lemma 1. The resulting 2012 lines together with ℓ_1 form the desired good arrangement.

For the other direction, let $\mathcal{P} = A_1 \cdots A_{4026}$ be a regular 4026-gon with A_i red for i odd and blue for i even. Form a Colombian configuration C with the vertices of $\mathcal{P}$ and another blue point B. We claim that any good arrangement for C has at least 2013 lines. Note that each of the 4026 pairs (A_i, A_{i+1}) of neighboring points contains points in different regions, so A_iA_{i+1} intersects at least one of the lines. Each line intersects $\mathcal{P}$ at most twice, so an arrangement with k lines can produce at most $2k$ intersections, implying that $2k \geq 4026$ and hence $k \geq 2013$, as desired.

This problem was proposed by Ivan Guo from Australia. The current formulation of this problem was suggested during IMO jury meetings by Leonardo I. M. Sandoval from Mexico.

3. Let ω be the circumcircle of ABC, and let O_1 be the circumcenter of $A_1B_1C_1$. Because A_1, B_1, and C_1 are on the boundary of ABC and O_1 is outside of ABC, $A_1B_1C_1$ is obtuse. Without loss of generality, assume that $\angle B_1A_1C_1$ is obtuse so that O_1 and A lie on the same side of line B_1C_1.

LEMMA 2. *The second intersection A_0 of ω and the circumcircle of triangle AB_1C_1 is the midpoint of arc $\widehat{BAC}$.*

Proof. By the definition of A_0, we have $\angle A_0BC_1 = \angle A_0BA = \angle A_0CA = \angle A_0CB_1$ and $\angle A_0C_1A = \angle A_0B_1A$; hence, AC_1B and AB_1C are similar. But $BC_1 = CB_1$, so these two triangles are congruent; hence, $A_0B = A_0C$. Because $AA_0B_1C_1$ is cyclic, we have $\angle C_1A_0B_1 = \angle C_1AB_1 = \angle BAC$, so A_0 lies on $\widehat{BAC}$ with $BA_0 = CA_0$, implying that A_0 is the midpoint of $\widehat{BAC}$. ■

By Lemma 2, a spiral similarity centered at A_0 sends B_1C_1 to CB, so A_0 is the intersection of ω and the perpendicular bisector of B_1C_1, which is on the same side of BC as A. Recalling that A_0 is the circumcenter of $A_1B_1C_1$, and using this result for the analogous points B_0 and C_0, we obtain that $A_0C_1B_0A_1$ and $A_0A_1C_0B_1$ are kites with symmetry axes A_0B_0 and A_0C_0. Recalling that $C_1B_1AA_0$ is cyclic, we have $\angle CAB = \angle C_1A_0B_1 = 2\angle B_0A_0C_0 = \widehat{B_0C_0}$. By Lemma 2, B_0 and C_0 are the midpoints of $\widehat{ABC}$ and $\widehat{BCA}$, hence

$$\angle CAB = \widehat{B_0C_0} = 360^\circ - \widehat{ACC_0} - \widehat{B_0A} = 360^\circ - \frac{\widehat{BCA} + \widehat{ABC}}{2}$$
$$= 360^\circ - \frac{360^\circ - 2\angle BCA + 360^\circ - 2\angle ABC}{2} = \angle BCA + \angle ABC,$$

implying that $\angle CAB = 90^\circ$, so ABC has right angle at vertex A.

This problem was proposed by Alexander Polyanskiy from Russia.

4. Let L be the foot of the altitude of ABC from A, and let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Because $\angle WNB < \angle CNB = 90^\circ$, O_1 and N lie on the same side of BC. Likewise, O_2 and M lie on the same side of BC. Hence, segment O_1O_2 does not intersect line BC. In particular, W does not lie on line O_1O_2, and ω_1 and ω_2 intersect again at a point Z other than W.

Because XW is a diameter of ω_1, we have $XZ \perp WZ$. Likewise, we have $YZ \perp WZ$, so X, Y, and Z lie on a line perpendicular to line ZW. It suffices to show that $HZ \perp ZW$. First, quadrilaterals $BNHL$ and $CMHL$ are cyclic, so by power of a point, $AM \cdot AC = AH \cdot AL = AN \cdot AB$; hence, A lies on the radical axis ZW of ω_1 and ω_2. Second, by our previous argument, A is the radical center of circles ω_1, ω_2, and the circumcircles of quadrilaterals $BNHL$ and $CMHL$. In particular, we have

$AH \cdot AL = AZ \cdot AW$, so $ZHLW$ is cyclic. This shows that $\angle AZH = \angle ALW = 90°$, hence $HZ \perp AW$, completing the proof.

This problem was proposed by Warut Suksompong and Potcharapol Suteparuk from Thailand.

5. We claim that the only solution is $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

By applying (i) and iterating (ii), we see that $f(n)f(x) \geq f(nx) \geq nf(x)$ for positive integer n. Setting $x = a$, we see that $f(n) \geq n$, because $f(a) = a > 0$.

We claim that f is non-negative and non-decreasing. Indeed, if $f(y) < 0$, setting $x = y$ and dividing by $f(y)$ in our first inequality shows that $f(n) \leq n$, hence $f(n) = n$. This chain of inequalities is thus an equality, so $f(n)f(x) = f(nx)$ for all x. Writing $y = \frac{p}{q} \in \mathbb{Q}_{>0}$, we have $f(q)f(\frac{p}{q}) = f(p)$, so $f(y) = y$, a contradiction. Hence, f is non-negative, thus also non-decreasing by (ii).

We claim now that $f(x) \geq x$ for all $x \geq 1$. First, note that $f(x) \geq f(\lfloor x \rfloor) \geq \lfloor x \rfloor > x - 1$. From (i) we know that $f(x)^n \geq f(x^n)$, so $f(x)^n \geq f(x^n) > x^n - 1$. But if $f(x) = x - \epsilon$ for some $\epsilon > 0$ and $x > 1$, then for all n we have $1 > x^n - f(x)^n \geq (x - f(x))(x^{n-1}) = \epsilon x^{n-1}$. Since $x > 1$, we can choose n such that $x^{n-1} > \frac{1}{\epsilon}$, a contradiction. Therefore, $f(x) \geq x$ for all $x > 1$, and we already know that $f(1) \geq 1$, yielding the claim.

We now show that $f(x) = x$ for $x \geq 1$. Note that $a^k = f(a)^k \geq f(a^k)$ for positive integers k by (i). We also have $f(a^k) \geq a^k$, so $f(a^k) = a^k$ for positive integers k. For $x \geq 1$ and k with $a^k > 2x$, we have $a^k = f(a^k) \geq f(x) + f(a^k - x) \geq x + (a^k - x) = a^k$. Equality thus holds, so $f(x) = x$ for $x \geq 1$.

Finally, for any integer n, we have $f(n) = n$ and $f(n)f(x) \geq f(nx) \geq nf(x)$, so equality holds, implying that $f(nx) = nf(x)$. In particular, for any $x = \frac{p}{q}$ in $\mathbb{Q}_{>0}$, we conclude that $qf(x) = f(p) = p$, hence $f(x) = x$, as desired.

This problem was proposed by Nikolai Nikolov from Bulgaria.

6. We will prove that there are $N + 1$ beautiful labelings for all $n \geq 2$.

Let $0 < x < 1$ be a real number. Define the beautiful labeling $C_n(x)$ as follows. For $0 \leq k \leq n$, let $W_k = e^{\frac{2\pi ik}{n}x}$, and let Z_k be the point that results from rearranging the W_k evenly in the same relative position; label Z_k by k. Note that W_aW_b and W_cW_d intersect iff Z_aZ_b and Z_cZ_d intersect.

Call such a labeling *cyclic*, and call it *degenerate* if two of the W_k coincide. Note that $C_n(x)$ is degenerate iff x is a reduced fraction with denominator at most n. Call such numbers *good*; there are N good fractions in $(0, 1)$.

LEMMA 3. *A labeling is beautiful if and only if it is non-degenerate cyclic.*

Proof. Let $C_n(x)$ be a non-degenerate cyclic labeling. For any $0 \leq a < b < c < d \leq n$ with $a + d = b + c$, arcs $\widehat{W_aW_b}$ and $\widehat{W_cW_d}$ have the same measure, so $W_aW_d \parallel W_bW_c$, implying that $C_n(x)$ is beautiful.

For the converse, induct on n with trivial base case $n = 2$. If all beautiful arrangements of $[0, n-1]$ are cyclic, for a beautiful arrangement A of $[0, n]$, form $A' = C_{n-1}(x)$ by removing n. Let x lie between consecutive good fractions p_1/q_1 and p_2/q_2 with $q_1, q_2 \leq n - 1$, and consider two cases.

Case 1: There is no fraction with denominator n between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. If $A \neq C_n(x)$, they can differ only in the location of n. Suppose W_n occurs directly between W_i and W_j in $C_n(x)$ in clockwise order. Note that $i + (n-1) = (i-1) + n$ and $j + (n-1) = (j-1) + n$, so the two corresponding pairs of chords do not intersect and $W_i, W_n, W_j, W_{i-1}, W_{n-1}, W_{j-1}$ occur in that order. In A, since W_iW_{n-1} does not intersect W_nW_{i-1}, W_n lies on arc $\widehat{W_iW_{n-1}}$. Likewise, W_n must lie on arc $\widehat{W_{n-1}W_j}$. Thus W_n lies between W_i and W_j, so $A = C_n(x)$.

Case 2: There is a fraction $\frac{a}{n}$ with denominator n between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. Since $\frac{p_2}{q_2} -$

$\frac{p_1}{q_1} \le \frac{1}{n-1}$, there is a unique such fraction. Choose $x_1 \in (\frac{p_1}{q_1}, \frac{a}{n})$ and $x_2 \in (\frac{a}{n}, \frac{p_2}{q_2})$. We wish to show that either $A = C_n(x_1)$ or $A = C_n(x_2)$. In A', W_{q_1}, W_0, W_{q_2} occur in that clockwise order. It suffices to show that W_n lies on arc $\widehat{W_{q_1}W_{q_2}}$ in A. This follows by an analysis of chords $W_{q_1}W_{n-1}$, $W_nW_{q_2-1}$, $W_{q_1-1}W_n$, and $W_{n-1}W_{q_2}$ using the final argument of Case 1. ■

By Lemma 3, it remains for us to count non-degenerate cyclic labelings. We claim that $C_n(x) = C_n(y)$ iff there is no good fraction between x and y. As x varies, the ordering of points in $C_n(x)$ changes only when $C_n(x)$ is degenerate so that two points coincide. It follows that $C_n(x) = C_n(y)$ when there is no good fraction between x and y. If there is a good fraction p/q with $x < p/q < y$, in $C_n(y)$ there are at least p integers $1 \le i \le q$ such that W_0 is clockwise of W_{i-1} and W_i is clockwise of W_0, while in $C_n(x)$ there are fewer than p such integers. Hence, $C_n(x)$ and $C_n(y)$ differ, giving the claim.

We conclude that the number of non-degenerate cyclic labelings is one greater than the number of good fractions in (0, 1), hence equal to $N + 1$.

This problem was proposed by Alexander Golovanov and Mikhail Ivaniv from Russia.

Results

The IMO was held in Santa Marta, Colombia, on July 23–24, 2013. There were 527 competitors from 97 countries and regions. On each day contestants were given four and a half hours for three problems.

The top score of 41/42 was shared by Yutao Liu (China) and Eunsoo Jee (South Korea). The USA team won 4 gold and 2 silver medals, placing third behind China and Korea. The students' individual results were as follows.

- Ray Li, who finished 12th grade at Phillips Exeter Academy in Exeter, NH, won a silver medal.
- Mark Sellke, who finished 11th grade at William Henry Harrison High School in West Lafayette, IN, won a gold medal.
- Bobby Shen, who finished 12th grade at Dulles High School in Sugar Land, TX, won a gold medal.
- Thomas Swayze, who finished 12th grade at Canyon Crest Academy in San Diego, CA, won a silver medal.
- James Tao, who finished 11th grade at Illinois Mathematics and Science Academy in Aurora, IL, won a gold medal.
- Victor Wang, who finished 12th grade at Ladue Horton Watkins High School in St. Louis, MO, won a gold medal.

2013 Carl B. Allendoerfer Awards

The Carl B. Allendoerfer Awards, established in 1976, are made to authors of expository articles published in *Mathematics Magazine*. The Awards are named for Carl B. Allendoerfer, a distinguished mathematician at the University of Washington and President of the Mathematical Association of America, 1959–60.

Khristo N. Boyadzhiev, "Close Encounters with the Stirling Numbers of the Second Kind," *Mathematics Magazine,* **85:4** (2012), pp. 252–266.

The Scottish mathematician James Stirling, in his 1730 book *Methodus Differentialis,* explored Newton series, which are expansions of functions in terms of difference polynomials. The coefficients of these polynomials, computed using finite differences, are the Stirling numbers of the second kind. Curiously, they arise in many other ways, ranging from scalar products of vectors of integer powers with vectors of binomial coefficients to polynomials that can be used to compute the derivatives of $\tan x$ and $\sec x$. This well-written exploration of Stirling numbers visits the work of Stirling, Newton, Grünert, Euler and Jacob Bernoulli. Boyadzhiev's fascinating historical survey centers on the representation of Stirling numbers of the second kind by a binomial transform formula. This might suggest a combinatorial approach to the study, but the article is novel in its analytical approach that mixes combinatorics and analysis. Grounded in Stirling's early work on Newton series, this analytical approach illustrates the value of considering alternatives to Taylor's series when expressing a function as a polynomial series. The story of Stirling numbers continues with the exponential polynomials of Johann Grünert and geometric polynomials in the works of Euler. Boyadzhiev shows the relation of Stirling numbers of the second kind to the Bernoulli numbers and Euler polynomials. The article closes with a brief look at Stirling numbers of the first kind, a nice touch that deftly brings the proceedings to a close. Boyadzhiev's lively exposition engages the reader and leaves one eager to learn more.

Response from Khristo N. Boyadzhiev. The Allendoerfer Award is an exciting milestone in my life. I am truly honored to be recognized by the Mathematical Association of America. With its mission and especially with its publishing operation the MAA unites and educates a vast mathematical community. *The Mathematics Magazine,* the *College Mathematics Journal,* the *Monthly,* and *Math Horizons* have become my good friends. Students and professors all over the world read them, discuss them, send materials, and work on the problem sections. What a treasure these journals are!

For my review on the Stirling numbers, I was inspired by the magic interplay between analysis and combinatorics. I was also inspired by the works of Henry W. Gould, Professor Emeritus at West Virginia University, who has a special taste for beautiful combinatorial identities. I am obliged to the editor Walter Stromquist for his friendly and competent help in bringing the manuscript to its final form. Thank you all!

Biographical Note

Khristo Boyadzhiev is a Professor of Mathematics at Ohio Northern University. He was born and educated in Sofia, Bulgaria. Khristo enrolled at Sofia University,

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"St. Kliment Ohridski," with the intention to study physics, but the calculus lectures of Yaroslav Tagamlitski changed his mind. Later Tagamlitski became his PhD advisor.

At the beginning of his career Khristo was interested mostly in Banach algebras and operator theory. Later in life he developed a steady interest in classical analysis.

He is married and has two daughters. In his spare time Khristo enjoys blogging, taking long walks around the beautiful ONU campus, and listening to classical music.

Adrian Rice and Ezra Brown, "Why Ellipses Are Not Elliptic Curves," *Mathematics Magazine,* **85:3** (2012), pp. 163–176.

While ellipses and elliptic curves are two topics most mathematicians know something about, few of us have considered how they relate to each other. It is clear that the equations of ellipses and elliptic curves are different, but why then are their names so similar? This excellent exposition explores where the related names came from despite the core differences in these two famous mathematical objects.

The authors begin with a brief history of ellipses, starting in Ancient Greece. We then quickly arrive at elliptic integrals, which first arose from the desire to compute the arc length of a section of an ellipse. This historical tour ends with Jacobi and Eisenstein working with the doubly periodic properties of elliptic functions, the inverses of elliptic integrals. This engaging article returns to the Greeks, with the focus on elliptic curves this time, then progresses to Fermat, goes on to Newton, and finally returns to Eisenstein to connect elliptic curves to elliptic functions. It is here that the two subjects come together; they are both connected, in their own way, to elliptic functions. The authors then give us the final twist: having parameterized elliptic curves using elliptic functions and ellipses using trigonometric functions, they show us that the parameterization of an ellipse in complex space is topologically a sphere, whereas the parameterization of an elliptic curve in complex space is topologically a torus.

Rice and Brown's well-written article weaves the story of these two diverse mathematical objects, giving key historical and mathematical references along the way. Their engaging tour of mathematical history illustrates both how these two objects are related and why, mathematically, they are fundamentally different.

Response from Ezra Brown and Adrian Rice. We are thrilled, honored, and deeply grateful to the Allendoerfer Committee for this award. This paper united our respective interests in elliptic curves (Bud) and elliptic functions (Adrian) with our joint interest in the history of mathematics, and it was great fun to write. In addition to the Committee, we wish to express our sincere thanks to Walter Stromquist for all his editorial support at the helm of *Mathematics Magazine,* and to our respective departments for their encouragement and interest in our work. Finally, we want to thank the MAA for three significant reasons: Firstly, meetings of the MAA's Maryland-DC-Virginia section fostered our initial collaboration. Secondly, it was an MAA journal that published the fruits of this partnership. And thirdly, it was a committee of the MAA that saw fit to honor our work in this way. In short, none of this would have been possible without the MAA. So thank you, MAA, once again!

Biographical Note

Ezra (Bud) Brown grew up in New Orleans, has degrees from Rice and Louisiana State University, and has been at Virginia Tech since 1969, where he is currently Alumni Distinguished Professor of Mathematics. His research interests include number theory and combinatorics, and elliptic curves have fascinated him for a long time.

He particularly enjoys discovering connections between apparently unrelated areas of mathematics and working with students who are engaged in research. He has been a frequent contributor to the MAA journals, and he recently served a term as the MD/DC/VA Section Governor. In his spare time, Bud enjoys singing (from opera to rock and roll), playing jazz piano, and solving word puzzles. Under the direction of his wife Jo, he has become a fairly tolerable gardener, and the two of them enjoy kayaking. He occasionally bakes biscuits for his students, and he once won a karaoke contest.

Adrian Rice received a B.Sc. in mathematics from University College London in 1992 and a Ph.D. in the history of mathematics from Middlesex University in 1997 for a dissertation on Augustus De Morgan. He is currently a Professor of Mathematics at Randolph-Macon College in Ashland, Virginia. His research focuses on nineteenth- and early twentieth-century mathematics, on which he has published research papers, articles and books, including *Mathematics Unbound: The Evolution of an International Mathematical Research Community, 1800–1945*, edited with Karen Hunger Parshall, *The London Mathematical Society Book of Presidents, 1865–1965*, written with Susan Oakes and Alan Pears, and *Mathematics in Victorian Britain*, edited with Raymond Flood and Robin Wilson. In his spare time, he enjoys reading, travel, and spending time with his wife and young son.

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CONTENTS

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS